Stats 579 – Intermediate Bayesian Modeling

Assignment # 4

1. Suppose y is a random variable with cdf $F(y) = 1 - e^{-\lambda y^\alpha}$ for $y > 0$, $\alpha > 0$. We say $y \sim \text{Weibull}(\lambda, \alpha)$. Explain how to simulate y from $\text{Uniform}(0, 1)$ random variables. Note that y is a simple transformation of an $\text{Exp}(\lambda)$ random variable. What is the transformation?

2. If $Y_i \overset{\text{iid}}{\sim} \text{Gamma}(a_i, b)$ for $i \in \{1, ..., k\}$, we have shown that

$$(Z_1, ..., Z_k) = \left(\frac{Y_1}{S}, ..., \frac{Y_k}{S}\right) \sim \text{Dirichlet}(a_1, ..., a_k),$$

   where $S = \sum_{i=1}^k Y_i$. Use Proposition B.4 to transform $Y_1, ..., Y_k$ into the random vector $Z_1, ..., Z_{k-1}, S$. Show that S is independent of the other variables by showing that the joint density of the random vector is the product of a $\text{Dirichlet}(a_1, ..., a_k)$ density and a $\text{Gamma}\left(\sum_{i=1}^k a_i, b\right)$ density. The general Dirichlet density is an obvious extension of the three-parameter Dirichlet density given in Table 2.1.

3. In acceptance-rejection sampling, consider the general choice of candidate distribution for a log-concave target density as presented in class.

   Recall that we have $\tilde{\theta}_1$ and $\tilde{\theta}_2$, points on either side of the mode of $\ell(\theta) \equiv \log[p_*(\theta)]$; and that we have tangent lines to $\ell(\theta)$ calculated at these points,

$$\gamma_i(\theta) = \ell(\tilde{\theta}_i) + \ell'(\tilde{\theta}_i)(\theta - \tilde{\theta}_i),$$

   where $i \in \{1, 2\}$. Then

$$p_*(\theta) = e^{\ell(\theta)} \leq e^{\gamma_i(\theta)}$$

   for both $i = 1$ and $i = 2$. We choose

$$Mq(\theta) = \min\left\{e^{\gamma_1(\theta)}, e^{\gamma_2(\theta)}\right\}.$$

   (a) Find θ_* by setting $\gamma_1(\theta_*) = \gamma_2(\theta_*)$ and solving.

   (b) Integrate $Mq(\theta)$ over $(-\infty, \infty)$ to determine M as a function of θ_*, $\tilde{\theta}_1$, and $\tilde{\theta}_2$.

   (c) Obtain the cdf based on the density $q(\theta)$, say

$$Q(v) \equiv \int_{-\infty}^{v} q(\theta)d\theta.$$

   Do this first for $v \leq \theta_*$ and then for $v > \theta_*$. Calculate the latter as $\int_{-\infty}^{\theta_*} q(\theta)d\theta + \int_{\theta_*}^{v} q(\theta)d\theta$.

   (d) Solve $Q(v) = u$ for v so that $v = Q^{-1}(u)$. Thus if we sample $U \sim \text{Uniform}(0, 1)$, we have $Q^{-1}(U) \sim q(\cdot)$.