1. An hypothetical study considers the lifespan of fluorescent light bulbs. Let $y_1, ..., y_n$ be the duration (in years) it takes for each of $n$ light bulbs to fail. Assume that all tests are performed under laboratory conditions and observations are iid. Researchers are interested in whether bulb lifespan is better modeled with model $M_1$, an Exp ($\lambda$) distribution, or with model $M_2$, a Weibull ($\alpha, \lambda$) distribution. For both models, the researchers assume $p(\lambda) = e^{-\lambda}$.

For this problem, please use the Exponential and Weibull parameterizations from your textbook, which give

$y_E \sim \text{Exp} (\lambda)$

$f(y_E | \lambda) = \lambda \exp (-\lambda y) I_{(0, \infty)}(y)$

$y_W \sim \text{Weibull} (\alpha, \lambda)$

$f(y_W | \alpha, \lambda) = \lambda \alpha y^{\alpha-1} \exp (-\lambda y^\alpha) I_{(0, \infty)}(y)$

(a) Obtain the marginal density for these data under each model. (HINT: Take advantage of conjugacy.)
(b) Obtain an expression for the Bayes factor comparing $M_1$ to $M_2$.
(c) Evaluate the Bayes factor when the data are: $\{8.05, 6.56, 3.20, 6.85, 5.67\}$.
(d) Explain which model seems preferable based on the Bayes factor. Explain which model would be preferable if you had a prior belief that $M_1$ were nine times more likely to be correct than $M_2$.

2. Using the same set-up as in Problem 1, the researchers want to compare these models in terms of AIC.

(a) Find the MLE for $\lambda$ under $M_1$ and $M_2$. Note that although both models have a parameter named $\lambda$, these parameters are not the same and may maximize at different values for each model. It may be helpful to write the MLEs as $\hat{\lambda}_1$ and $\hat{\lambda}_2$ to help distinguish them.
(b) Calculate AIC for each model using the data provided above. Which model seems preferable based on AIC?

3. Let $y_1, ..., y_n$ be independent conditional on some model parameter $\theta$. Let $y_i \sim f(y_i | \theta)$ and let $\theta$ have prior $p(\theta)$. Consider the conditional predictive ordinate for an observation $y_j$,

$$CPO_j = f(y_j | y_{(j)}),$$

where $y_{(j)}$ denotes the set $\{y_1, ..., y_{j-1}, y_{j+1}, ..., y_n\}$.

(a) Show that

$$CPO_j = \frac{\int \prod_{i=1}^{n} f(y_i | \theta) p(\theta)}{\int \prod_{i \neq j} f(y_i | \theta) p(\theta)}.$$

(b) Now show that

$$CPO_j^{-1} = \int \left[ \frac{1}{f(y_j | \theta)} \right] p(\theta | y) d\theta.$$