Stats 579 – Intermediate Bayesian Modeling

Assignment # 1

1. Let $y_i|\theta \overset{\text{iid}}{\sim} \text{Bern}(\theta)$ for $i \in \{1, ..., n\}$, and let $p(\theta) = I_{[0,1]}(\theta)$, i.e. θ is uniform on $[0, 1]$.
   (a) Obtain the marginal density of $(y_1, ..., y_n)$.
   (b) Calculate the predictive probability that $y_{n+1} = 1$ given that $y_1 = ... = y_n = 1$. Simplify the formula you get using the fact that $\Gamma(a+1) = a\Gamma(a)$ and thus establish that for $n = 1000$, $\Pr[y_{n+1} = 1] = \frac{1001}{1002}$.

2. Suppose in a random sample of 10 transportation workers, all were found to be on drugs. Find Box's marginal p-value and perform Bayesian significance tests to evaluate whether such data are consistent with the following models:
   (a) $y_1, ..., y_{10} \mid \theta \overset{\text{iid}}{\sim} \text{Bern}(0.10)$
   (b) $y_1, ..., y_{10} \mid \theta \overset{\text{iid}}{\sim} \text{Bern}(\theta)$; $\theta \sim \text{Beta}(0.05, 0.45)$
   (c) $y_1, ..., y_{10} \mid \theta \overset{\text{iid}}{\sim} \text{Bern}(\theta)$; $\theta \sim \text{Beta}(1, 1)$

3. Let $y|\theta \sim \text{Pois}(\theta)$. Conduct the following Bayesian hypothesis tests:
   (a) Assume a prior on θ of $\text{Gamma}(1, 1)$. For $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$ obtain the formula for the posterior probability that H_0 is true. Use software (e.g. R, WinBUGS) to calculate the probability for $y = 3$, 5, and 7.
   (b) For $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$ with $q_0 = 0.5$ and $p_1(\theta) = e^{-\theta}$, obtain the analytical formula for the posterior probability that H_0 is true. Use software (e.g. WinBUGS, R) to calculate the exact probabilities for $y = 3$, 5, and 7.

4. For $h > 0$, let $x_1, ..., x_h$ be exchangeable random quantities and let

$$y_h = \frac{1}{h} \sum_{i=1}^{h} x_i.$$

   Assume that for each random quantity x_i, all moments exist—that is,

$$\text{E}[x_i] = \mu_1, \quad \text{E}\left[x_i^2\right] = \mu_2, \quad ....$$

   Further assume that $n \leq h$ and let j_i for $i \in \{1, ..., h\}$ represent any of the $h!$ re-orderings of the indicies $1, ..., h$. Then

$$\text{E}\left[\prod_{i=1}^{n} x_{j_i}\right] = m_n.$$

   Prove that for all n,

$$\lim_{h\to\infty} \text{E}[y_h^n] = m_n$$