Elements of Mathematical Statistics
Homework Assignment #1

Due 20 February 2020 by the start of class.

Sample Spaces. For each of the following experiments, (1) detail the outcomes contained in the *sample space* (S) of the experiment, (2) determine whether each sample space is *discrete* or *continuous* and (3) give an example of a non-trivial event A (i.e. $A \neq S$ and $A \neq \emptyset$).

a) Experiment: Drawing a card at random from a standard deck of 52 cards.

b) Experiment: Measuring the time it takes for Katie Ledecky to swim the 400m Freestyle.

c) Experiment: Counting how many times a lightbulb can be switched on before it burns out.

Venn Diagrams. For each part below, create a 3-set Venn Diagram (with a border representing S) and shade in the region corresponding to the given event.

a) $(A \cup B)^c$

b) $(A \cap B) \cup C$

c) $(A \cup B \cup C) \cap (A \cap B \cap C)^c$

Probability Rules. Mike is about to start watching "Game of Thrones" and "Westworld". The probability that Mike likes "Game of Thrones" is 0.4. The probability that Mike likes "Westworld" is 0.5. The probability that Mike likes both shows is 0.2. Define the events

$$A = \{\text{Mike likes Game of Thrones}\} \qquad B = \{\text{Mike likes Westworld}\}$$

a) Are A and B independent? Are they disjoint? Justify your answers mathematically.

b) Find the probability that Mike doesn't like Game of Thrones or Westworld.

c) Find the probability that Mike likes Westworld, given that he doesn't like Game of Thrones.

Pairs of Socks. Noted traveler Rick Steves has 10 red socks, 8 blue socks and 12 black socks. He is packing for vacation, and hastily throws 6 socks into his suitcase.

a) What is the probability that Rick has selected a pair of each color?

a) What is the probability that Rick has selected three pairs of matching socks, regardless of color?

Jar of Coins. *Note: This is adapted from a Google interview question.* I have a jar with 100 quarters. Most of them are regular (fair) quarters, but 15 of them are fake! Of these fake quarters, 10 have heads on both sides and 5 have tails on both sides.

a) If you select a coin from the jar at random and flip it 3 times and get heads every time, what is the probability that this is a "two-heads" quarter?

b) Repeat the calculation in part a) but this time you flip the coin k times, getting heads each time. What is the probability that this is a "two-heads" quarter (in terms of k)?

c) Using a computer, plot this probability as a function of k. Based on your plot, for what value of k is the probability more than $1/2$? For what value of k is the probability more than 0.9?

Birthday Paradox. Suppose n people are at a party.

a) What is the probability that at least two people share a birthday? (*Hint: It is easier to first look at the complement of this event.*) Using a computer, create a plot showing this probability as a function of n and attach it to your homework.

b) How many people need to be in the room before this probability is greater than 1/2? Is this surprising?

c) *Pigeonhole Principle.* How many people need to be in the room before this probability is 1? Explain the intuition behind this.

Challenge Problem. **The Dating Problem.** Alf is ready to find a wife and settle down, so naturally, he turns to Tinder. Alf has N matches, each of which can be ranked from 1 to N. His goal is to find the best possible match with high probability. He decides to use the following probabilistic dating scheme.

- He will date each of his matches sequentially (in a completely random order). At the end of the date, he will make an *immediate* and *irrevocable* decision on whether or not to propose. (Yeah, Alf moves fast).
- For some integer $x > 1$, he will automatically reject the first $x - 1$ matches. He will then propose to the i^{th} match only if i^{th} date was better than all of the previous $i - 1$ dates.

a) Let $x = 2$ and find the probability that Alf finds his soulmate, i.e. propses to the best possible match. (*Hint: Let A be the event that Alf proposes to the best match, and let B_i be the event that the i^{th} date is the best one. Use Law of Total Probability.*)

b) Generalize your answer for any value of x less than N.

c) When N is large, the following identity holds (approximately).

$$\frac{x-1}{N}\sum_{i=x}^{N}\frac{1}{i-1} \simeq -\frac{x}{N}\ell n(x/N)$$

Using this approximation, show (using calculus) that the choice $x = N/e$ maximizes the probability of hiring the best candidate. If Alf has $N = 100$ matches, how many dates should he go on before he starts thinking about proposing? What is the probability that he finds his soulmate?