Stationarity

- This is a very important concept in T.S. Most of the theory in T.S. is based on stationarity.
- For *state space models*, we will relax this assumption.
- We have two types of stationarity.
- **Second order stationarity:** The mean is constant in time and the covariance is a function of the difference in time between observations

$$E(X_t) = \mu; \quad Cov(X_t, X_s) = f(|t - s|)$$

- **Autocovariance function:** For a second order stationary process, this is defined as

$$Cov(X_t, X_{t+h}) = Cov(X_t, X_{t-h}) = \gamma(h)$$

- **Strict Stationarity:** The probability distribution of $X_{t_1}, X_{t_2}, \ldots X_{t_n}$ is invariant for any time shift of size δ

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = p(x_{t_1+\delta}, x_{t_2+\delta}, \ldots, x_{t_n+\delta})$$

- Strict stationarity implies second order stationarity. For $n = 1$, $p(x_t) = p(x_{t+\delta})$, then $E(X_t) = \mu$. For $n = 2$, $p(x_t, x_s) = p(x_{t+h}, x_{s+h})$
- The reverse statement is not true except when we have a *Gaussian* process X_t

Autocovariance and autocorrelation

- The concepts of autocovariance and autocorrelation are key to formulate time series models for stationary processes.
- The main utility of these functions, is that they can help us identify a time series process.
- We'll discuss these points further when we introduce ARMA models.
- As before, the theoretical autocovariance of a (second order or weak) stationary process X_t is defined as:

$$\gamma(h) = cov\{X_t, X_{t+h}\} = E(X_t - \mu)(X_{t+h} - \mu)$$

- μ is the overall mean of X_t

- h is a positive integer that represents the "hth-lag".
- By extension, the theoretical autocorrelation function of X_t is defined in terms of the autocovariance as:

$$\rho(h) = \gamma(h)/\gamma(0)$$

- This implies that $\rho(h)$ is between -1 and 1.
- Alternatively and since we are assumning stationarity, we could replace X_{t+h} by X_{t-h} in the above definitions.
- $\gamma(h)$ and $\rho(h)$ are measures of the dependency between observations separated by h units of time.
- Based on a realization x_t of a time series process, to estimate the theoretical autocovariance function we use

$$g_h = \sum_{t=h+1}^{n} (x_t - \overline{x})(x_{t-h} - \overline{x})/n; \quad h = 0, 1, \ldots$$

- $\overline{x}$ is the sample mean.
- To estimate the autocorrelation function at lag h, we use

$$r_h = g_h/g_0; \quad h = 0, 1, \ldots$$

- It is important to realize that g_h is only formed with $n - h$ terms divided into n.
- If we use $n - h$ as a divisor instead of n, we could finish with estimated autocorrelations (i.e. values of r_h) that are greater than one or less than -1.
- To visualize the dependencies of x_t for different lags h, we use the *Correlogram*.

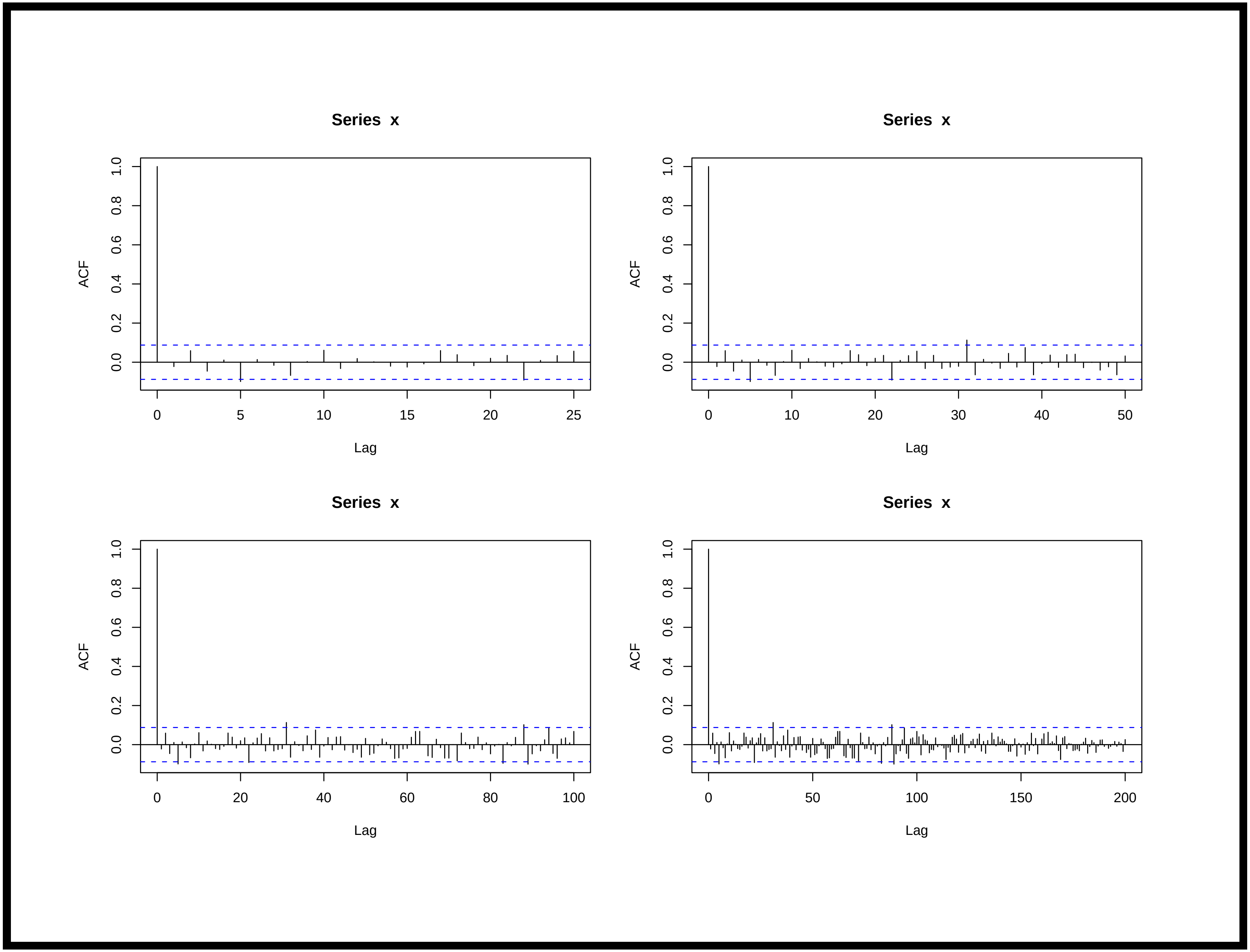
- A correlogram is a plot of h (x-axis) versus its corresponding value of r_h (y-axis).
- The correlogram may exhibit patterns and different degrees of dependency in a time series.
- A "band"of size $2/\sqrt{n}$ is added to the correlogram because asymptotically $r_h \sim N(0, 1/n)$ if the data is close to a white noise process.
- This band is used to detect *significant* autocorrelations, i.e. autocorelations that are different from zero.

Example

- 500 simulated observations of a white noise process (N(0,1)).
- Using the function *acf* in R, we may plot the autocorrelation function for different lags.

```
x <- rnorm(500)
par(mfrow=c(2,2))
lg <- c(25,50,100,200)
for(i in 1:4){acf(x,lag=lg[i])}
mtext("Correlogram for white noise data at
different lags",outer=T)
```

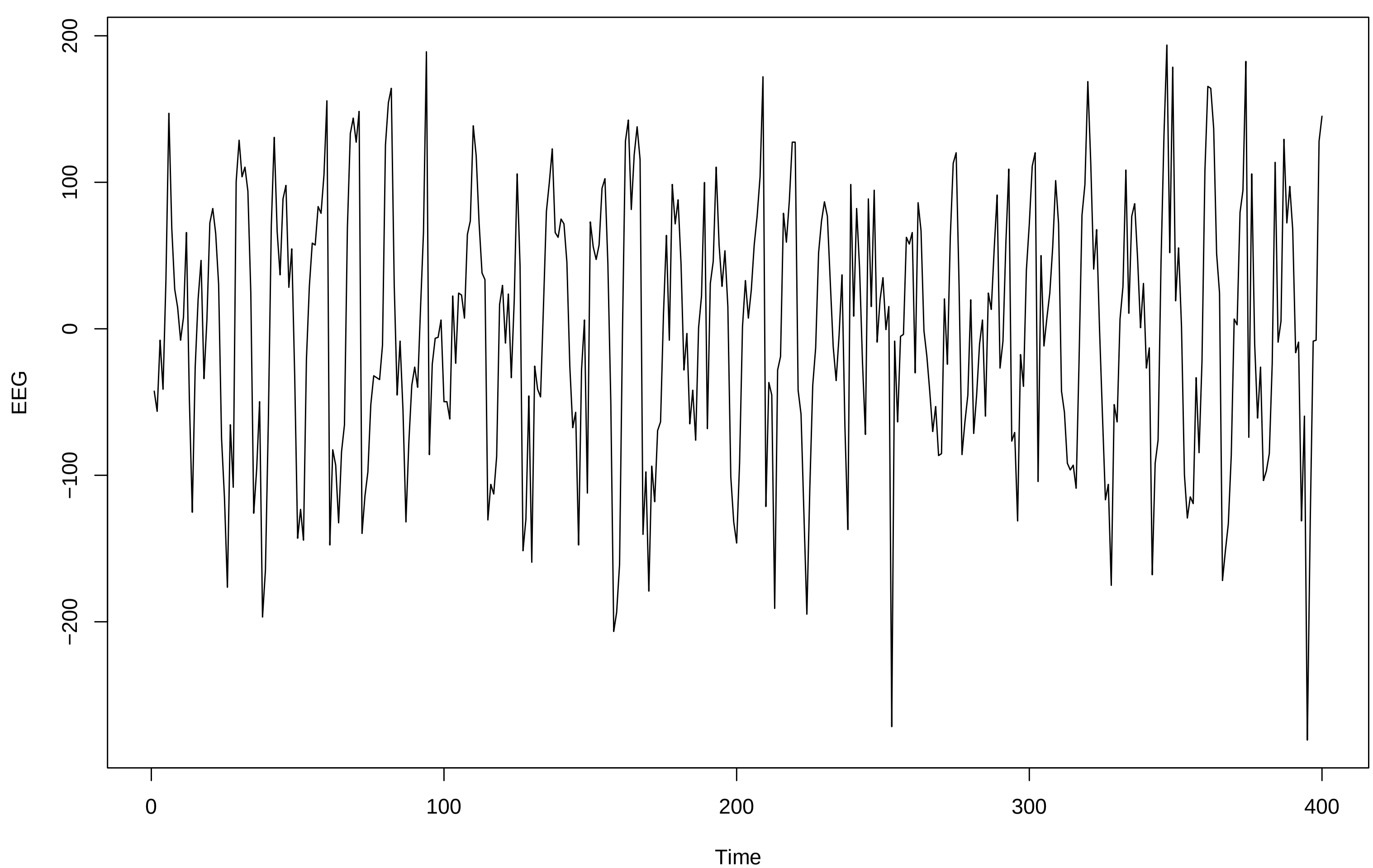
- notice that ocassionally we will have autocorrelations outside the confidence band.


Series x
ACF
Lag
Series x
ACF
Lag
Series x
ACF
Lag
Series x
ACF
Lag

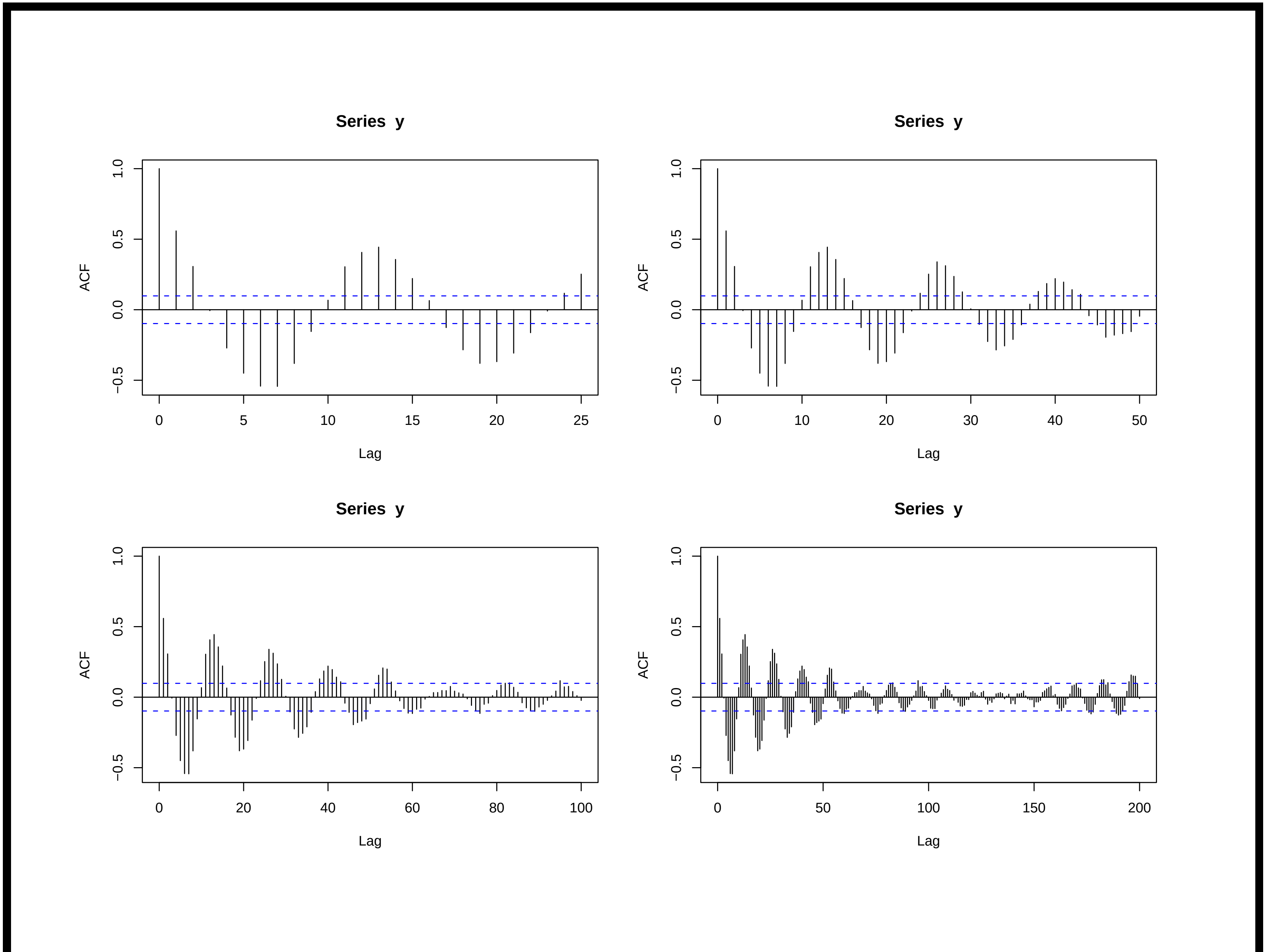
Example

- 400 EEG recordings.
- First we will see the time series plot.
- Picture of the ACF was obtained for lags 25,50,100 and 200.

```
y <- scan("eeg")
y <- as.ts(y)
plot(y,ylab="EEG")
par(mfrow=c(2,2))
lg <- c(25,50,100,200)
for(i in 1:4){acf(y,lag=lg[i])}
mtext("Correlograms for EEG",outer=T)
```



EEG
200
100
0
−100
−200
0
100
200
300
400
Time


Series y
ACF
Lag
Series y
ACF
Lag
Series y
ACF
Lag
Series y
ACF
Lag

- Notice that in this last example, the correlogram exhibits cyclic variation.
- This ACF cyclic variation is the same as for the time series.
- For example, with monthly data if r_6 is 'large' and positive then r_{12} is 'large' and negative.
- If x_t follows a sinusoidal pattern, r_h also follows this pattern.
- For example, if $x_t = A\ cos(\omega t)$ where A is a constant, ω is frequency where $0 < \omega < \pi$,

$$r_h \approx cos(h\omega)$$