# Derivation of Conditional Density

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**Abstract**

KEY WORDS:

Suppose

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \right)$$

and let $V_{1\cdot 2} = V_{11} - V_{12}V_{22}^{-1}V_{21}$. We show that

$$Y_1|Y_2 \sim N\left(\mu_1 + V_{12}V_{22}^{-1}[Y_2 - \mu_2], V_{1\cdot 2}\right),$$

assuming that the distributions are nonsingular.

To show the result, we need to verify that the density for the stated conditional distribution is the ratio of the joint density to the marginal density of $Y_2$. This requires simplification of the terms in the exponents of the densities, which is what we focus on. It also requires knowing that with

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

we have

$$\det(V) = \det(V_{22})\det(V_{1\cdot 2}).$$

We start with $-2$ times the exponent of the conditional density and show that it is $-2$ times the difference between the joint and marginal density exponents. Without loss of generality, assume the mean vectors are 0. Recall (PA, Appendix B) that

$$V^{-1} = \begin{bmatrix} V_{1\cdot 2}^{-1} & -V_{1\cdot 2}^{-1}V_{12}V_{22}^{-1} \\ \\ -V_{22}^{-1}V_{21}V_{1\cdot 2}^{-1} & V_{22}^{-1} + V_{22}^{-1}V_{21}V_{1\cdot 2}^{-1}V_{12}V_{22}^{-1} \end{bmatrix}.$$

For the conditional density

$$\begin{aligned} &\left(Y_1 - V_{12}V_{22}^{-1}Y_2\right)' V_{1\cdot 2}^{-1} \left(Y_1 - V_{12}V_{22}^{-1}Y_2\right) \\ &= \begin{bmatrix} Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} V_{1\cdot 2}^{-1} & -V_{1\cdot 2}^{-1}V_{12}V_{22}^{-1} \\ \\ -V_{22}^{-1}V_{21}V_{1\cdot 2}^{-1} & V_{22}^{-1}V_{21}V_{1\cdot 2}^{-1}V_{12}V_{22}^{-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} V_{1\cdot 2}^{-1} & -V_{1\cdot 2}^{-1} V_{12} V_{22}^{-1} \\ -V_{22}^{-1} V_{21} V_{1\cdot 2}^{-1} & V_{22}^{-1} + V_{22}^{-1} V_{21} V_{1\cdot 2}^{-1} V_{12} V_{22}^{-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - Y_2' V_{22}^{-1} Y_2 \\
&= \begin{bmatrix} Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - Y_2' V_{22}^{-1} Y_2.
\end{aligned}
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