FREE RESOLUTIONS OF MONOMIAL IDEALS

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Abstract. In this thesis, we explore the construction and properties of free resolutions of monomial ideals in polynomial rings. We focus on the Taylor resolution, one of the earliest algorithmic approaches to constructing free resolutions, alongside the general construction for a minimal graded free resolution examine its minimality. Specifically, we give examples of minimal and nonminimal Taylor resolutions, identify conditions under which the Taylor resolution is minimal, and highlight the impact of redundant least common multiples in generating scalar entries in the resolution's matrices. In addition, we provide a general form for the minimal free resolution of any bivariate monomial ideal in a polynomial ring $F[x, y]$. We also study the Taylor resolution compared to its truncated subcomplex, the Lyubeznik resolution, and discuss some computational methods for obtaining resolutions. This work reiterates insights into the minimality of free resolutions and provides concrete examples to demonstrate these concepts.

1. Introduction

I first encountered the content discussed in this thesis at the Diversity in the Mathematical Sciences 2024 summer school on Combinatorial Commutative Algebra at Dalhousie University. Over the course of a week, I attended lectures and labs instructed by Nasrin Altafi, Selvi Kara, Sarah Mayes-Tang, Susan Morey, and Mayada Shahada. The notes I took at this summer school have been a valuable resource that I have consistently referenced while working on this thesis [AKMT$^+$].

There are multiple ways to construct an exact sequence of free modules to form a free resolution of the quotient module R/I for some ideal I of R. In particular, monomial ideals in polynomial rings over fields are given by explicit generators, the relations of which are represented in the syzygies of the resolutions.

In this thesis, we study and compare constructions of free resolutions of monomial ideals, such as those from [Tay66], [Lyu88], and [Pee11]. A free resolution of a module informs us of the generators of the syzygies, which arise from the kernel of each map in the exact sequence of module homomorphisms. The study of a free resolution can also inform us about the projective dimension

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of a module. Furthermore, a monomial ideal serves as the initial ideal for some polynomial ideals in the polynomial ring, so the resolution of a monomial ideal also helps develop a division algorithm for multivariate polynomial ideals.

This thesis also examines properties of resolutions and differentiating characteristics such as minimality. The length of the minimal free resolution gives the projective dimension of a module, so obtaining a minimal resolution from a convenient but highly nonminimal construction is a key aim. We consider properties of the monomial generators that indicate when a construction will produce a minimal resolution, particularly for the Taylor resolution.

The minimality of a free resolution is directly tied to the ranks of the modules and the Betti numbers. In the construction of the Taylor resolution, modules in the resolution are spanned by basis elements associated with subsets of the generators of the ideal. Then, the least common multiple (lcm) of these subsets of generators are taken, and may yield redundant lcm's in the set. This redundancy in least common multiples manifests as scalar entries in the presentation matrices, which do not appear in minimal resolutions. We utilize this observation to give a condition for when the Taylor resolution of a monomial ideal is minimal. Finally, this thesis provides a general form for the minimal free resolution of any bivariate monomial ideal in a polynomial ring $F[x, y]$. The form is a simple construction and shows that the Taylor resolution of a monomial ideal in two variables is minimal only when the ideal is generated by two or fewer monomials. A more extensive cataloging of resolutions of monomial ideals was carried out in recent work [CK24], which considers the Barile-Macchia resolutions. This thesis only briefly mentions the Barile-Macchia resolution construction, given by [BM20], as a contrast to the Lyubeznik construction we consider.

2. Background and Definitions

This thesis provides extensive background information so that our discussion of resolutions may be accessible those not yet acquainted with abstract algebra.

Definition 2.1. [DF04, Definition 1 of Chapter 1] *A group is a set G with a binary operation $\star : G \times G \to G$ satisfying:*

(1) $\star$ is associative; that is, $a \star (b \star c) = (a \star b) \star c$ for all $a, b, c \in G$.
(2) There exists an identity element $i \in G$ such that $i \star x = x \star i = x$ for all $x \in G$.
(3) Every element $a \in G$ has an inverse element $a' \in G$ such that $a \star a' = a' \star a = i$.

A subset $H \subseteq G$ which is also a group under the same operation is called a subgroup.

The following proposition gives a sufficient condition to determine whether a subset of a group is a subgroup.

Proposition 2.2. [DF04, Exercise 26 of section 1.1] *A nonempty subset H of a group is a subgroup if and only if for all $a, b \in H$, $a \star b' \in H$, where b' denotes the inverse of b.*

The following example utilizes the above proposition to show that a subset of matrices in the group of $n \times n$ invertible matrices is a subgroup.

Example 2.3. *For fixed $n \in \mathbb{Z}^+$, the set of invertible $n \times n$ matrices with entries in the real numbers, denoted $GL_n(\mathbb{R})$, forms a group under matrix multiplication. For $n = 2$, we show that the set of upper triangular invertible 2×2 matrices with entries in $\mathbb{R}$, in this example denoted H, is a subgroup of $GL_2(\mathbb{R})$.*

Let $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in H$. *We want to show that* $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^{-1} \in H$

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^{-1} = \frac{1}{xz} \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}.$$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \frac{1}{xz} \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix} = \frac{1}{xz} \begin{pmatrix} a \cdot z + b \cdot 0 & a \cdot (-y) + b \cdot x \\ 0 \cdot z + c \cdot 0 & 0 \cdot (-y) + c \cdot x \end{pmatrix} = \begin{pmatrix} \frac{a}{x} & \frac{bx-ay}{xz} \\ 0 & \frac{c}{z} \end{pmatrix} \in H$$

The resulting matrix is also an upper triangular matrix with entries in $\mathbb{R}$, so H is a subgroup of $GL_n(\mathbb{R})$.

Definition 2.4. [DF04, Definition 1 of Chapter 7] *A set R equipped with two binary operations, addition and multiplication, is a ring if the following are satisfied:*

(1) $(R, +)$ *is an abelian group; that is,* $r + s = s + r$ *for any* $r, s \in R$*.*
(2) Multiplication is associative on R; that is, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for every* $a, b, c \in R$
(3) Multiplication is distributive over addition; that is, $a \cdot (b+c) = a \cdot b + a \cdot c$ *and* $(a+b) \cdot c = a \cdot c + b \cdot c$ *for every* $a, b, c \in R$*.*

We say a ring R is commutative if its multiplication commutes. i.e., $a \cdot b = b \cdot a$ *for all* $a, b \in R$*. A subring of R is a subset under addition which is a subgroup and closed under multiplication.*

In this thesis, R will always refer to a commtative ring and F will always refer to a field as given in the following definition.

Definition 2.5. [DF04, Definition 2 of Chapter 7] *A field is a commutative ring F with its multiplication satisfying the following:*

(1) Multiplication is commutative on F.
(2) Every nonzero element of F has a multiplicative inverse; that is, if $x \in F$ *and* $x \neq 0$*, then there exists* $x^{-1} \in F$ *such that* $x \cdot x^{-1} = 1$*.*

The kind of ring we primarily deal with is a polynomial ring $F[x_1, \ldots, x_n]$ over some field F. In the following definition and in this thesis, we take $\mathbb{N}$ to include 0, which impacts the following definition by allowing that some powers of variables be zero.

$$F[x_1, \ldots, x_n] = \left\{ \sum_{(a_1, \ldots, a_n) \in \mathbb{N}^n} c_{(a_1, \ldots, a_n)} x_1^{a_1} \cdots x_n^{a_n} \mid \text{only finitely many } c_{(a_1, \ldots, a_n)} \in F \text{ are nonzero} \right\}$$

Elements of $F[x_1, \ldots, x_n]$ are called polynomials and a polynomial with only one term $x_1^{a_1} \cdots x_n^{a_n}$ is called a monomial. Henceforth we may denote vectors of indeterminates by $\mathbf{x} = (x_1, \ldots, x_n)$ and exponent vectors by $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,n})$ so that $\mathbf{x}^{\mathbf{a}_i} = x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}$.
Multiplication and addition are defined on $F[x_1, \ldots, x_n]$ as follows.
Let $f = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ and $g = \sum_{\mathbf{a} \in \mathbb{N}^n} d_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in F[x_1, \ldots, x_n]$.

$$f + g := \sum_{\mathbf{a} \in \mathbb{N}^{\ltimes}} (c_{\mathbf{a}} + d_{\mathbf{a}}) \mathbf{x}^{\mathbf{a}}.$$

$$f \cdot g := \sum_{\mathbf{a} \in \mathbb{N}^k} \Big(\sum_{\mathbf{q}, \mathbf{r} \in \mathbb{N}^n | q_i + r_i = a_i} c_{\mathbf{q}} d_{\mathbf{r}} \Big) \mathbf{x}^{\mathbf{a}}.$$

Note that every polynomial is a finite sum, hence the condition that only finitely many coefficients are nonzero, but expressing a polynomial as a sum over every possible exponent vector allows us to define multiplication and addition of polynomials without having to consider the number of terms.

The following definitions introduce a grading on a polynomial ring $F[x_1, \dots, x_n]$ over field F.

Definition 2.6. [Pee11, 1.1] *For each $i = 1, \dots, k$, set $deg(x_i) = 1$. The degree of a monomial is given by $\deg(x_1^{a_1} \cdots x_n^{a_n}) = a_1 + \cdots + a_k$. A polynomial $f = \sum_{(a_1, \dots, a_n) \in \mathbb{N}^n} c_{(a_1, \dots, a_n)} x_1^{a_1} \cdots x_n^{a_n}$ is called homogeneous if for every nonzero $c_{(a_1, \dots, a_n)}$, the sum $a_1 + \cdots + a_n$ is the same.*

Definition 2.7. [DF04, Definition 4 of Chapter 7] *A commutative ring R with multiplicative identity is called an integral domain if for any $a, b \in R$ with $ab = 0$ then $a = 0$ or $b = 0$*

Note that any field is an integral domain, so the following proposition applies to our case of a polynomial ring over a field.

Proposition 2.8. [DF04, Proposition 1 of Chapter 9] *Let R be an integral domain. Then*

(1) For any nonzero $f, g \in R[x_1, \dots, x_n]$, $\deg(f \cdot g) = \deg(f) + \deg(g)$,
(2) The units of $R[x_1, \dots, x_n]$ are precisely the units of R,
(3) $R[x_1, \dots, x_n]$ is an integral domain.

Definition 2.9. [Pee11] *If $R = F[x_1, \dots, x_n]$, then denote by R_i the F-module spanned by all monomials of degree i. That is, R_i consists of all F-linear combinations of monomials of total degree i.*

Every polynomial $f \in R = F[x_1, \dots, x_n]$ can be written uniquely as a finite sum $f = \sum_i f_i$ of non-zero homogeneous components $f_i \in R_i$. Thus, the polynomial ring can be expressed as the direct sum

$$R = \bigoplus_{i \in \mathbb{N}} R_i = \left\{ \sum_{i \in \mathbb{N}}^{\text{finite}} c_i f_i \mid c_i \in F, f_i \in R_i \text{ for } i \in \mathbb{N} \right\}$$

Definition 2.10. [DF04, Definition 7 of Chapter 7] *Let R be a commutative ring and $I \subseteq R$. Say I is an ideal of R if I is an additive subgroup of R and for every $r \in R$ and every $x \in I$, we have $rx \in I$.*

We will primarily be dealing with ideals generated by monomials, so given a monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n} \in F[x_1, \ldots, x_n]$, the ideal generated by that monomial is $(\mathbf{x}^{\mathbf{a}}) = \{\mathbf{x}^{\mathbf{a}} \cdot f \mid f \in F[x_1, \ldots, x_n]\}$.

Proposition 2.11. *The ideal generated by the set of monomials* $S = \{\boldsymbol{x}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{a}_k}\}$ *is*

$$(S) = \{\boldsymbol{x}^{\boldsymbol{a}_1} \cdot f_1 + \cdots + \boldsymbol{x}^{\boldsymbol{a}_k} \cdot f_k \mid f_1, \ldots, f_k \in F[x_1, \ldots, x_n]\}$$

Proof. Let's verify the claim that the set $\{\mathbf{x}^{\mathbf{a}_1} \cdot f_1 + \cdots \mathbf{x}^{\mathbf{a}_k} \cdot f_k \mid f_1, \ldots, f_k \in F[x_1, \ldots, x_n]\}$, denoted by I in this proof, is an ideal.
First, let $\mathbf{x}^{\mathbf{a}_1} \cdot f_1 + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot f_k, \mathbf{x}^{\mathbf{a}_1} \cdot g_1 + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot g_k \in I$ for some $f_1, \ldots, f_k, g_1, \ldots, g_k \in F[x_1, \ldots, x_n]$ $\mathbf{x}^{\mathbf{a}_1} \cdot f_1 + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot f_k - (\mathbf{x}^{\mathbf{a}_1} \cdot g_1 + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot g_k) = \mathbf{x}^{\mathbf{a}_1} \cdot (f_1 - g_1) + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot (f_k - g_k)$ is also an element of I because $f_j - g_j \in F[x_1, \ldots, x_n]$ for $j = 1, \ldots, k$, so I is a subgroup of $F[x_1, \ldots, x_n]$.
Now let $\mathbf{x}^{\mathbf{a}_1} \cdot f_1 + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot f_k \in I$ and $g \in F[x_1, \ldots, x_n]$. $(\mathbf{x}^{\mathbf{a}_1} \cdot f_1 + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot f_k) \cdot g = \mathbf{x}^{\mathbf{a}_1} \cdot (f_1 \cdot g) + \cdots + \mathbf{x}^{\mathbf{a}_k} \cdot (f_k \cdot g)$ is also an element of I because $f_j \cdot g \in F[x_1, \ldots, x_n]$ for $j = 1, \ldots, k$.
This concludes the proof that I is an ideal. □

A monomial is called squarefree if every exponent a_j of x_j is either 0 or 1. Given n indeterminates in $F[x_1, \ldots, x_n]$, a squarefree monomial has n exponents, each of which is either 0 or 1. Thus, there are 2^n distinct squarefree monomials with coefficient 1. Furthermore, there are at most 2^{2^n} ideals generated by squarefree monomials in $F[x_1, \ldots, x_n]$. We see a pivotal example in section 3 of resolutions of a squarefree monomial ideal which demonstrate the distinction between two free resolution constructions this thesis discusses, the Lyubeznik resolution and the minimal graded resolution.

Example 2.12. *Consider the polynomial ring* $F[x, y]$. *Find all squarefree monomials and squarefree monomial ideals. The squarefree monomials are* $\{1, x, y, xy\}$.
The nontrivial squarefree monomial ideals are $(1) = F[x, y]$, *the ring itself which is generated by* 1; (x) *and* (y), *the sets of polynomials which are multiples of* x *and* y, *respectively;* (xy), *the set of polynomials which are multiples of* xy; *and* (x, y), *the set of polynomials for which every term has a factor of* x *or* y.
Those are all the distinct squarefree monomial ideals in $F[x, y]$. *Any other combination of squarefree monomial generators of an ideal would produce a redundant ideal. Any ideal with* 1 *as a generator is the entire ring, and the following equivalencies are obvious:* $(xy, x) = (x)$, $(xy, y) = (y)$, *and* $(xy, x, y) = (x, y)$.

Definition 2.13. [DF04, Definition 1 of Chapter 10] *Let R be a ring and M a set with a binary operation $+$ on M such that $(M, +)$ is an abelian group. Call M an R-module if there is an action of R on M, $\cdot : R \times M \to M$, which satisfies the following:*

(1) $(r + s) \cdot m = r \cdot m + s \cdot m$ for every $r, s \in R$ and $m \in M$.
(2) $(rs) \cdot m = r \cdot (s \cdot m)$ for every $r, s \in R$ and $m \in M$.
(3) $r \cdot (m + n) = r \cdot m + r \cdot n$ for every $r \in R$ and $m, n \in M$.

If R has a multiplicative identity 1, then we also have that $1 \circ m = m$ for every $m \in M$.

Modules generalize the familiar concept of vector spaces. An R-module when R is a field is called a vector space. Though the structures are similar, some properties of vector spaces are not applicable to modules over rings because a ring need not be commutative nor have multiplicative inverses for each element. For instance, every finitely generated vector space has a basis but there exist finitely generated modules over rings which do not have a basis.

Definition 2.14. [DF04, Definition 2 of Chapter 10] *A nonempty subset N of an R-module M is an R-submodule of M if N is a subgroup of M and is closed under the action of ring elements.*

A submodule N of an R-module M is also an R-module. N inherits its commutativity from M so it is also an abelian group and if N is closed under the action then the three conditions on the action are satisfied again by inheritence from M.

Example 2.15. *The polynomial ring $F[x_1 \ldots, x_n]$ introduced above is an F-vector space as well as a ring. To show $F[x_1 \ldots, x_n]$ is a vector space, first show that it is an abelian group. The identity of $F[x_1 \ldots, x_n]$ is 0. For $f = \sum_{\boldsymbol{a} \in \mathbb{N}^n} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$, $-f = \sum_{\boldsymbol{a} \in \mathbb{N}^n} (-c_{\boldsymbol{a}}) \boldsymbol{x}^{\boldsymbol{a}}$*
Let $f = \sum_{\boldsymbol{a} \in \mathbb{N}^n} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}, g = \sum_{\boldsymbol{a} \in \mathbb{N}^n} d_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$ be polynomials in $F[x_1, \ldots, x_n]$.

$$f + g = \sum_{\boldsymbol{a} \in \mathbb{N}^n} (c_{\boldsymbol{a}} + d_{\boldsymbol{a}}) \boldsymbol{x}^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathbb{N}^n} (d_{\boldsymbol{a}} + c_{\boldsymbol{a}}) \boldsymbol{x}^{\boldsymbol{a}} = g + f$$

because F being a field entails that F is an Abelian group, so $c_{\boldsymbol{a}} + d_{\boldsymbol{a}} = d_{\boldsymbol{a}} + c_{\boldsymbol{a}}$ for $\forall c_{\boldsymbol{a}}, d_{\boldsymbol{a}} \in R$. So $F[x_1 \ldots, x_n]$ is an Abelian group.

The action of F on $F[x_1 \dots, x_n]$ is defined by $r \cdot (c_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \dots + c_k \boldsymbol{a}^{\boldsymbol{a}_k}) = (rc_1)\boldsymbol{x}^{\boldsymbol{a}_1} + \dots + (rc_k)\boldsymbol{x}^{\boldsymbol{a}_k}$ *where rc_j is multiplication in F. To show that this action satisfies the conditions of Definition 2.13. let $r, s \in F$ and* $f = \sum_{\boldsymbol{a} \in \mathbb{N}^n} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}, g = \sum_{\boldsymbol{a} \in \mathbb{N}^n} d_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}} \in F[x_1, \dots, x_n]$

For (1),

$$(r+s) \cdot f = (r+s)c_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots (r+s)c_k \boldsymbol{x}^{\boldsymbol{a}_k} = (rc_1 + sc_1)\boldsymbol{(}x)^{\boldsymbol{a}_1} + \cdots + (rc_k + sc_k)\boldsymbol{x}^{\boldsymbol{a}_k}$$

$$= rc_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots rc_k \boldsymbol{x}^{\boldsymbol{a}_k} + sc_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots sc_k \boldsymbol{x}^{\boldsymbol{a}_k} = r \cdot f + s \cdot f.$$

For (2),

$$(rs) \cdot f = (rs)c_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots (rs)c_k \boldsymbol{x}^{\boldsymbol{a}_k} = r(sc_1)\boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + r(sc_k)\boldsymbol{x}^{\boldsymbol{a}_k} = r \cdot (s \cdot f).$$

For (3),

$$r \cdot (f+g) = r \cdot \sum_{\boldsymbol{a} \in \mathbb{N}^n} (c_{\boldsymbol{a}} + d_{\boldsymbol{a}})\boldsymbol{x}^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathbb{N}^n} r(c_{\boldsymbol{a}} + d_{\boldsymbol{a}})\boldsymbol{x}^{\boldsymbol{a}}$$

$$= \sum_{\boldsymbol{a} \in \mathbb{N}^n} (rc_a)\boldsymbol{x}^{\boldsymbol{a}} + \sum_{\boldsymbol{a} \in \mathbb{N}^n} (rd_a)\boldsymbol{x}^{\boldsymbol{a}} = r \cdot \sum_{\boldsymbol{a} \in \mathbb{N}^n} (c_a)\boldsymbol{x}^{\boldsymbol{a}} + r \cdot \sum_{\boldsymbol{a} \in \mathbb{N}^n} (d_a)\boldsymbol{x}^{\boldsymbol{a}} = r \cdot f + r \cdot g$$

Definition 2.16. [DF04, Definition 4 of Chapter 10] *Let M and N be two R-modules. A function $f : M \to N$ is called an R-module homomorphism if for every $x, y \in M$, $f(x+y) = f(x) + f(y)$ and for every $r \in R$, $f(rx) = rf(x)$.*
Define the kernel of a homomorphism $f : M \to N$ to be the set $\ker(f) = \{x \in M \mid f(x) = 0\}$ *where* 0 *is the additive identity of N.*

Lemma 2.17. [DF04, Theorem 4 of Chapter 10] *Given an R-module homomorphism $f : M \to N$,* $\ker(f)$ *is a submodule of M and $f[M]$, the image of f, is a submodule of N.*

Proof. The claims of the lemma are part of larger claims in the First Isomorphism Theorem for Modules, so we provide a precise proof of these statements. To show that these two sets are submodules, we show that each is closed under subtraction and closed under the action of ring elements.
Let $a, b \in \ker(f)$, so $f(a) = f(b) = 0$. $f(a-b) = f(a) - f(b) = 0$, so $(a-b) \in \ker(f)$. Let

$a \in \ker(f), r \in R$. $f(r \cdot a) = r \cdot f(a) = r \cdot 0 = 0$, so $r \cdot a \in ker(f)$.

Let $f(x), f(y) \in f[M]$ for some $x, y \in M$. $f(x) - f(y) = f(x - y)$ is an element of the image of f because $(x - y) \in M$. Let $f(x) \in f[M], r \in R$. $r \cdot f(x) = f(r \cdot x)$ is an element of the image of f because $r \cdot x \in M$. □

Definition 2.18. [DF04, Definition 8 of Chapter 10] *Say an R-module M is free on a subset $A \subset M$ if every $m \in M$ can be written uniquely up to reordering as $m = r_1a_1 + \cdots + r_na_n$ for some $n \in \mathbb{Z}^+$, $r_i \in R$, $a_i \in M$ for $i = 1, \ldots, n$. In this case, say A is a basis for M.*

Every F-vector space V is a free module and every basis of V has the same cardinality.

Definition 2.19. [DF04, Definition 8 of Chapter 10] *The rank of a free R-module M is equal to the cardinality of a basis A of M. Analogously, the dimension of an F-vector space V is equal to the cardinality of a basis for V.*

Definition 2.20. [Pee11, Section 2] *An R-module M is called graded if it has a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ as an F-vector space satisfying that $\{r_im_j \mid r_i \in R_i \text{ and } m_j \in M_j\} = R_iM_j$ is contained in M_{i+j} for every $i, j \in \mathbb{N}$. Call M_i a homogeneous component of M.*

Every polynomial $f \in F[x_1, \ldots, x_n]$ can be expressed as the sum of finitely many nonzero homogeneous components, $f = \sum_{i\in\mathbb{N}} f_i$ where each summand f_i is in the homogeneous component $F[x_1, \ldots, x_n]_i$ of the polynomial ring.

Definition 2.21. [Pee11, Section 1] *An ideal I of a graded polynomial ring $F[x_1, \ldots, x_n]$ is called graded or homogeneous if it satisfies the two following equivalent conditions:*

(1) I has a system of homogeneous generators.

(2) If a polynomial $f \in I$, then each homogeneous component of f is in I.

Proof. Here we prove the claim that the two conditions in Definition 2.21 are indeed equivalent. If $\{g_\alpha\}_{\alpha\in\mathcal{A}}$ generates I where each g_α is a polynomial in degree $c_\alpha \in \mathbb{N}$, then any $f \in I$ has $f = \sum_{\alpha\in\mathcal{A}} g_\alpha h_\alpha$ for some homogeneous polynomials h_α. Then each term $g_\alpha h_\alpha$ is homogeneous of degree $\deg(g_\alpha)+\deg(h_\alpha)$. Then the degree d homogeneous component of f is $\sum_{\substack{\alpha\in\mathcal{A}\\ \deg(g_\alpha)+\deg(h_\alpha)=d}} g_\alpha h_\alpha$. Then clearly the homogeneous component is generated by $\{g_\alpha\}_{\alpha\in\mathcal{A}}$ and belongs to the ideal.

Now assume that for any polynomial $f \in I$, its degree d homogeneous component also belongs to I. I claim that I is generated by the set $S = \{g \in I \mid g \text{ is homogeneous}\}$ Clearly, $(S) \subseteq I$, so want to show $I \subseteq (S)$. Let $f \in I$, so each homogeneous component, f_d the component of degree d, is an element of the ideal and of S. So $f \in (S)$ because $f = \sum_{d\in\mathbb{N}} f_d$. □

We introduce an ordering on monomials $\mathbf{x}^{\mathbf{a_1}}, \ldots, \mathbf{x}^{\mathbf{a_k}} \in F[x_1, \ldots, x_n]$ by

$$x_1^{a_{1,1}} \cdots x_n^{a_{1,n}} < x_1^{a_{2,1}} \cdots x_n^{a_{2,n}} \text{ if } a_{1,1} + \cdots + a_{1,n} < a_{2,1} + \cdots + a_{2,n}, \tag{1}$$

If $a_{1,1} + \cdots + a_{1,n} = a_{2,1} + \cdots + a_{2,n}$ but $(a_{1,1}, \ldots, a_{1,n}) \neq (a_{2,1}, \ldots, a_{2,n})$, then there exists $j = \min(i \mid a_{1,i} \neq a_{2,i})$. In this case, where $p, q \in \{1, 2\}$ are distinct, $a_{p,j} < a_{q,j}$ implies $\mathbf{x}^{\mathbf{a}_p} < \mathbf{x}^{\mathbf{a}_q}$

To manipulate the gradings on modules, we introduce a shifting that will impact the resolutions. For $p \in \mathbb{Z}$, set $M(-p) = \bigoplus_{i\in\mathbb{Z}} M_{i-p}$ and call this the module M shifted p degrees.

Let $M_1, \ldots, M_k$ be a collection of R-modules. The collection of k-tuples $(m_1, \ldots, m_k)$ where each $m_i \in M_i$ is called the direct product, denoted $M_1 \times \cdots \times M_k$. The direct product $M_1 \times \cdots \times M_k$ is also an R-module with addition and the action of R defined componentwise.

Proposition 2.22. *If M and N are free R-modules with respective ranks m and n, then multiplication by an $n \times m$ matrix with elements in R is an R-module homomorphism.*

Proof. Denote by A an $n \times m$ matrix $\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & & \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$ with each $a_{i,j} \in R$ for $i = 1, \dots, n, j = 1, \dots, m$. For $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in M$, $A\mathbf{x} = \begin{pmatrix} a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + \dots + a_{1,m} \cdot x_m \\ a_{2,1} \cdot x_1 + a_{2,2} \cdot x_2 + \dots + a_{2,m} \cdot x_m \\ \vdots \\ a_{n,1} \cdot x_1 + a_{n,2} \cdot x_2 + \dots + a_{n,m} \cdot x_m \end{pmatrix} \in N$.

Let $\mathbf{x}, \mathbf{y} \in M$ and $r \in R$.

$$\begin{aligned}
A(\mathbf{x}+\mathbf{y}) = A \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_m + y_m \end{pmatrix} &= \begin{pmatrix} a_{1,1} \cdot (x_1 + y_1) + a_{1,2} \cdot (x_2 + y_2) + \dots + a_{1,m} \cdot (x_m + y_m) \\ a_{2,1} \cdot (x_1 + y_1) + a_{2,2} \cdot (x_2 + y_2) + \dots + a_{2,m} \cdot (x_m + y_m) \\ \vdots \\ a_{n,1} \cdot (x_1 + y_1) + a_{n,2} \cdot (x_2 + y_2) + \dots + a_{n,m} \cdot (x_m + y_m) \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} \cdot x_1 + a_{1,1} \cdot y_1 + a_{1,2} \cdot x_2 + a_{1,2} \cdot y_2 + \dots + a_{1,m} \cdot x_m + a_{1,m} \cdot y_m \\ a_{2,1} \cdot x_1 + a_{2,1} \cdot y_1 + a_{2,2} \cdot x_2 + a_{2,2} \cdot y_2 + \dots + a_{2,m} \cdot x_m + a_{2,m} \cdot y_m \\ \vdots \\ a_{n,1} \cdot x_1 + a_{n,1} \cdot y_1 + a_{n,2} \cdot x_2 + a_{n,2} \cdot y_2 + \dots + a_{n,m} \cdot x_m + a_{n,m} \cdot y_m \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + \dots + a_{1,m} \cdot x_m \\ a_{2,1} \cdot x_1 + a_{2,2} \cdot x_2 + \dots + a_{2,m} \cdot x_m \\ \vdots \\ a_{n,1} \cdot x_1 + a_{n,2} \cdot x_2 + \dots + a_{n,m} \cdot x_m \end{pmatrix} + \begin{pmatrix} a_{1,1} \cdot y_1 + a_{1,2} \cdot y_2 + \dots + a_{1,m} \cdot y_m \\ a_{2,1} \cdot y_1 + a_{2,2} \cdot y_2 + \dots + a_{2,m} \cdot y_m \\ \vdots \\ a_{n,1} \cdot y_1 + a_{n,2} \cdot y_2 + \dots + a_{n,m} \cdot y_m \end{pmatrix} \\
&= A\mathbf{x} + A\mathbf{y}.
\end{aligned}$$

$$
\begin{aligned}
A(r\cdot\mathbf{x}) &= A\begin{pmatrix} r\cdot x_1 \\ r\cdot x_2 \\ \vdots \\ r\cdot x_m \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}\cdot(r\cdot x_1)+a_{1,2}\cdot(r\cdot x_2)+\cdots+a_{1,m}\cdot(r\cdot x_m) \\ a_{2,1}\cdot(r\cdot x_1)+a_{2,2}\cdot(r\cdot x_2)+\cdots+a_{2,m}\cdot(r\cdot x_m) \\ \vdots \\ a_{n,1}\cdot(r\cdot x_1)+a_{n,2}\cdot(r\cdot x_2)+\cdots+a_{n,m}\cdot(r\cdot x_m) \end{pmatrix} \\
&= \begin{pmatrix} r\cdot(a_{1,1}\cdot x_1)+r\cdot(a_{1,2}\cdot x_2)+\cdots+r\cdot(a_{1,m}\cdot x_m) \\ r\cdot(a_{2,1}\cdot x_1)+r\cdot(a_{2,2}\cdot x_2)+\cdots+r\cdot(a_{2,m}\cdot x_m) \\ \vdots \\ r\cdot(a_{n,1}\cdot x_1)+r\cdot(a_{n,2}\cdot x_2)+\cdots+r\cdot(a_{n,m}\cdot x_m) \end{pmatrix} \\
&= r\begin{pmatrix} a_{1,1}\cdot x_1+a_{1,2}\cdot x_2+\cdots+a_{1,m}\cdot x_m \\ a_{2,1}\cdot x_1+a_{2,2}\cdot x_2+\cdots+a_{2,m}\cdot x_m \\ \vdots \\ a_{n,1}\cdot x_1+a_{n,2}\cdot x_2+\cdots+a_{n,m}\cdot x_m \end{pmatrix} \\
&= rA\mathbf{x}.
\end{aligned}
$$

□

3. Resolutions

Definition 3.1. [DF04, Definition 1 of section 10.5] *Let R be a ring and $M_1, \dots, M_n$ be R-modules with $f_i : M_i \to M_{i+1}$ homomorphisms for $i = 1, \dots, n-1$ such that we have a sequence*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$$

We say the sequence is exact at M_j if $\operatorname{im}(f_j) = \ker(f_{j+1})$.

A free resolution of an R-module M is an exact sequence

$$\cdots \longrightarrow F_m \xrightarrow{\partial_m} F_{m-1} \longrightarrow \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

in which each F_i is a free R-module. Such a resolution of a monomial ideal is our object of study. The actual module of which we take the resolution is R/I where $R = F[x_1, \ldots, x_n]$ and I is the ideal generated by some monomials $m_1, \ldots, m_r$. The ε map will be the natural projection $\pi : R \to R/I$ defined by $\pi(f) = f + I$. The kernel of that map is the ideal $I \subset R$, so the map ∂_1 can be given in terms of the generators of I. The next map ∂_2 contains information on the relations between generators of the ideal.

Definition 3.2. [AKMT$^+$] *Call a free resolution of a module minimal if every entry of the presentation matrices of the maps lies in the maximal ideal* $< x_1, \ldots, x_n >$ *of* $F[x_1, \ldots, x_n]$.

Definition 3.3. [DF04, Definition 3 of section 9.6]

(1) *The leading term of a nonzero polynomial* $f \in F[x_1, \ldots, x_n]$*, denoted* $LT(f)$*, is the monomial term of maximal order in* f*, according to the monomial ordering* $<$ *of Equation 1.*

(2) *If* I *is an ideal of* $F[x_1, \ldots, x_n]$*, the ideal of leading terms, denoted* $LT(I)$*, is the ideal generated by the leading terms of all the elements of the ideal. i.e.,* $LT(I) =< LT(f) \mid f \in I >$.

Definition 3.4. [DF04, Definition 4 of section 9.6] *A Gröbner basis for an ideal* I *in a polynomial ring* $F[x_1, \ldots, x_n]$ *is a finite set of generators* $\{g_1, \ldots, g_m\}$ *for* I *whose leading terms generate the ideal of leading terms of* I*. i.e.,*

$$I =< g_1, \ldots, g_m > \text{ and } LT(I) =< LT(g_1), \ldots, LT(g_m) >$$

There is no division algorithm in a polynomial ring with more than one variable. Given an ideal in $F[x_1, \ldots, x_n]$, the Gröbner basis gives a set of monomials by which one can divide polynomials in the ideal. LT(I) is usually called the initial ideal of I, as in [Gre10].

There is a more general idea of a resolution of a module called a projective resolution in which each F_i must be a projective R-module and not necessarily free. An R-module is projective if it is a direct summand of a free R-module [DF04, Definition 20 of Chapter 10] This is a more general definition than that of a free resolution because all free modules are projective. The property of projectivity is important in studying the set of homomorphisms between the projective module and other modules and other aspects about the module. The projective dimension of a module M is defined to be the length of the shortest possible projective resolution which can be constructed of

M. If a module M is projective, then the projective dimension of M is 0 because

$$M \longrightarrow 0$$

is a projective resolution and has length 0. In this way, the projective dimension of a module measures how close the module is to being projective.

Unsurprisingly, minimal free resolutions provide the most useful information toward the projective dimension of a module. The difference between a nonminimal and a minimal free resolution of a module can be the omission of multiple modules in multiple homological steps or even removing whole homological steps themselves. The difference between minimal and nonminimal resolutions can be seen in the comparison between the nonminimal Taylor resolution construction and the inherently minimal construction of a minimal graded free resolution

The first construction for a free resolution of a monomial ideal that we examine is the Taylor resolution. The Taylor was the first algorithmic approach to finding free resolutions of monomial ideals.

Taylor Resolution Construction: [Tay66]

Given a monomial ideal I in a polynomial ring R minimally generated by monomials $m_1, \dots, m_r$, define $F_i(I) = \{\ [m_j]_{j \in J}\} \mid |J| = i$ and $\{m_j\}_{j\in J} \subseteq \{m_1, \dots, m_r\}\}$ for $i = 1, \dots, r$.
Then define $R^{F_i(I)}$ to be the R-module $\{\sum_{|J|=i} r_J[m_j]_{j\in J} \mid r_J \in R\}$, which is isomorphic to $R^{|F_i(I)|}$ [AKMT$^+$]. In the resolution, the modules will be presented as R^k where $k = |F_i(I)|$ in the resolution but it is helpful to view $F_i(I)$ as a basis for the module in homological degree i, $R^{F_i(I)}$ as given above.
Define $\partial_i : R^{F_i(I)} \to R^{F_{i-1}(I)}$ for $i = 1, \dots, r$ as follows:

$$\partial_i([m_j]_{j\in J}) = \sum_{k\in J} (-1)^{\varepsilon(J,k)} \frac{\operatorname{lcm}(m_j)_{j\in J}}{\operatorname{lcm}(m_j)_{j\in J\setminus\{k\}}}$$

Where $\varepsilon(J,k) = |\{j \in J \mid j < k\}|$.

∂_1 can be represented by the matrix $\begin{pmatrix} m_1 \dots m_r \end{pmatrix}$.

∂_2 can be represented by the matrix

$$\begin{pmatrix} \frac{\mathrm{lcm}(m_1,m_2)}{m_1} & \frac{\mathrm{lcm}(m_1,m_3)}{m_1} & \dots & \frac{\mathrm{lcm}(m_1,m_r)}{m_1} & 0 & \dots & 0 \\ -\frac{\mathrm{lcm}(m_1,m_2)}{m_2} & 0 & \dots & 0 & \frac{\mathrm{lcm}(m_2,m_3)}{m_2} & \dots & 0 \\ 0 & -\frac{\mathrm{lcm}(m_1,m_3)}{m_3} & \dots & 0 & -\frac{\mathrm{lcm}(m_2,m_3)}{m_3} & \dots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{\mathrm{lcm}(m_{r-1},m_r)}{m_{r-1}} \\ 0 & 0 & \dots & -\frac{\mathrm{lcm}(m_1,m_r)}{m_r} & 0 & \dots & -\frac{\mathrm{lcm}(m_{r-1},m_r)}{m_r} \end{pmatrix}$$

While the matrix for ∂_3 and maps in higher homological degrees can be constructed similarly, they become increasingly cumbersome to write out explicitly, making it difficult to represent them in a general matrix form.

Example 3.5. *Construct the Taylor Resolution of $I =< x^2, xy, y^3, z^2 >$ in $R[x, y, z]$. Since this ideal is generated by four monomials, the resolution will have the following form:*

$$0 \longrightarrow R \xrightarrow{\partial_4} R^4 \xrightarrow{\partial_3} R^6 \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R \longrightarrow 0$$

∂_1 is represented by the matrix $\begin{pmatrix} x^2 & xy & z^2 & y^3 \end{pmatrix}$ whose entries are the generators of the ideal. By this definition, the image of ∂_1 is the ideal $< x^2, xy, z^2, y^3 >\subseteq R$

Following the formula given above, ∂_2 is represented by the matrix

$$\begin{pmatrix} \frac{\mathrm{lcm}(x^2,xy)}{x^2} & \frac{\mathrm{lcm}(x^2,z^2)}{x^2} & \frac{\mathrm{lcm}(x^2,y^3)}{x^2} & 0 & 0 & 0 \\ -\frac{\mathrm{lcm}(x^2,xy)}{xy} & 0 & 0 & \frac{\mathrm{lcm}(xy,z^2)}{xy} & \frac{\mathrm{lcm}(xy,y^3)}{xy} & 0 \\ 0 & -\frac{\mathrm{lcm}(x^2,z^2)}{z^2} & 0 & -\frac{\mathrm{lcm}(xy,z^2)}{z^2} & 0 & \frac{\mathrm{lcm}(z^2,y^3)}{z^2} \\ 0 & 0 & -\frac{\mathrm{lcm}(x^2,y^3)}{y^3} & 0 & -\frac{\mathrm{lcm}(xy,y^3)}{y^3} & -\frac{\mathrm{lcm}(z^2,y^3)}{y^3} \end{pmatrix}$$

$$= \begin{pmatrix} y & z^2 & y^3 & 0 & 0 & 0 \\ -x & 0 & 0 & z^2 & y^2 & 0 \\ 0 & -x^2 & 0 & -xy & 0 & y^3 \\ 0 & 0 & -x^2 & 0 & -x & -z^2 \end{pmatrix}$$

In the same way, ∂_3 is given by

$$\begin{pmatrix} \frac{\operatorname{lcm}(x^2,xy,z^2)}{\operatorname{lcm}(x^2,xy)} & \frac{\operatorname{lcm}(x^2,xy,y^3)}{\operatorname{lcm}(x^2,xy)} & 0 & 0 \\ -\frac{\operatorname{lcm}(x^2,xy,z^2)}{\operatorname{lcm}(x^2,z^2)} & 0 & \frac{\operatorname{lcm}(x^2,z^2,y^3)}{\operatorname{lcm}(x^2,z^2)} & 0 \\ 0 & -\frac{\operatorname{lcm}(x^2,xy,y^3)}{\operatorname{lcm}(x^2,y^3)} & -\frac{\operatorname{lcm}(x^2,z^2,y^3)}{\operatorname{lcm}(x^2,y^3)} & 0 \\ \frac{\operatorname{lcm}(x^2,xy,z^2)}{\operatorname{lcm}(xy,z^2)} & 0 & 0 & \frac{\operatorname{lcm}(xy,z^2,y^3)}{\operatorname{lcm}(xy,z^2)} \\ 0 & \frac{\operatorname{lcm}(x^2,xy,y^3)}{\operatorname{lcm}(xy,y^3)} & 0 & -\frac{\operatorname{lcm}(xy,z^2,y^3)}{\operatorname{lcm}(xy,y^3)} \\ 0 & 0 & \frac{\operatorname{lcm}(x^2,z^2,y^3)}{\operatorname{lcm}(z^2,y^3)} & \frac{\operatorname{lcm}(xy,z^2,y^3)}{\operatorname{lcm}(z^2,y^3)} \end{pmatrix} = \begin{pmatrix} z^2 & y^2 & 0 & 0 \\ -y & 0 & y^3 & 0 \\ 0 & -1 & -z^2 & 0 \\ x & 0 & 0 & y^2 \\ 0 & x & 0 & -z^2 \\ 0 & 0 & x^2 & x \end{pmatrix}$$

And ∂_4 by

$$\begin{pmatrix} \frac{\operatorname{lcm}(x^2,xy,y^3,z^2)}{\operatorname{lcm}(x^2,xy,z^2)} \\ -\frac{\operatorname{lcm}(x^2,xy,y^3,z^2)}{\operatorname{lcm}(x^2,xy,y^3)} \\ \frac{\operatorname{lcm}(x^2,xy,y^3,z^2)}{\operatorname{lcm}(x^2,z^2,y^3)} \\ -\frac{\operatorname{lcm}(x^2,xy,y^3,z^2)}{\operatorname{lcm}(xy,z^2,y^3)} \end{pmatrix} = \begin{pmatrix} y^2 \\ -z^2 \\ 1 \\ -x \end{pmatrix}$$

The resulting Taylor resolution of $I =< x^2, xy, z^2, y^3 >$ is

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y^2 \\ -z^2 \\ 1 \\ -x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} z^2 & y^2 & 0 & 0 \\ -y & 0 & y^3 & 0 \\ 0 & -1 & -z^2 & 0 \\ x & 0 & 0 & y^2 \\ 0 & x & 0 & -z^2 \\ 0 & 0 & x^2 & x \end{pmatrix}}$$

$$R^6 \xrightarrow{\begin{pmatrix} y & z^2 & y^3 & 0 & 0 & 0 \\ -x & 0 & 0 & z^2 & y^2 & 0 \\ 0 & -x^2 & 0 & -xy & 0 & y^3 \\ 0 & 0 & -x^2 & 0 & -x & -z^2 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x^2 & xy & z^2 & y^3 \end{pmatrix}} R$$

This resolution is not minimal because in multiple maps there is an entry of the presentation matrix not in the maximal ideal and the columns of the presentation maps of ∂_2 and ∂_3 have columns which are linearly dependent. We can attempt to minimize the Taylor resolution.

We now discuss the construction for a minimal graded free resolution of a monomial ideal and compare the two constructions.

Minimal Graded Free Resolution Construction: [Pee11, Construction 4.2]
Given a monomial ideal I in a polynomial ring $R = F[x_1, \ldots, x_n]$ minimally generated by monomials $m_1, \ldots, m_r$ with respective degrees $a_1, \ldots, a_r$, we construct a minimal graded free resolution of R/I. We do this by induction on the module's position in the resolution, which we call the homological degree of the free module.
Step 0: The 0th map of the resolution will be the projection of $F_0 = R$ into $U_0 = R/I$, so take $\partial_0 : R \to R/I$, $\partial_0(f) = f + I$.
Step 1: Set $F_1 = R(-a_1) \bigoplus \cdots \bigoplus R(-a_{r_1})$ and $U_1 = \ker \partial_0 = I \subseteq R = F_0$. Define $\partial_1 : F_1 \to U_1$ by $\partial_1(f_1, \ldots, f_r) = \begin{pmatrix} m_1 \ldots m_r \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$.
Step $i+1$: Assume by induction that F_i and ∂_i are defined with ∂_i given as an $n \times m$ matrix. Set $U_{i+1} = \ker \partial_i$ and choose minimal homogeneous generators $l_1, \ldots, l_s \in \mathbb{R}^m$ with respective degrees $b_1, \ldots, b_{r_{i+1}}$ to be the columns of the presentation matrix. "Minimal" in this sense means that each generator l_j is an element of the maximal ideal $< x_1, \ldots, x_n >$ of R, so omitting any generator which has constant entries. We call the position in the resolution, denoted by i in the construction, the homological degree of F_i.

The resulting minimal graded free resolution of R/I will have the form

$$0 \to R(-a_{i,1}) \bigoplus \cdots \bigoplus R(-a_{i,r_i}) \xrightarrow{\partial_i} \cdots$$

$$\xrightarrow{\partial_3} R(-a_{2,1}) \bigoplus \cdots \bigoplus R(-a_{2,r_2}) \xrightarrow{\partial_2} R(-a_{1,1}) \bigoplus \cdots \bigoplus R(-a_{1,r_1}) \xrightarrow{\partial_1} R \xrightarrow{\partial_0} R/I$$

Definition 3.6. *Given the graded minimal free resolution above, where we define* $F_i = \bigoplus_{p \in \mathbb{Z}} R(-p)^{c_{i,p}}$ *for only finitely many nonzero* $c_{i,p}$*s, define the* i*th Betti number to be* $c_i := \sum_{p \in \mathbb{Z}} c_{i,p}$*. Say* $c_{i,p}$ *has homological degree* i *and internal degree* p*.*

Because there are finitely many modules in the resolution, the number of nonzero Betti numbers is finite. We represent the graded Betti numbers of a resolution using a Betti table where finitely many entries are nonzero. The top row of the table denotes the ith homological degree. The left-hand column labels the internal degree p, but the entry in the pth row and ith column is $c_{i,i+p}$.

	0	1	2	...
0	$c_{0,0}$	$c_{1,1}$	$c_{2,2}$	...
1	$c_{0,1}$	$c_{1,2}$	$c_{2,3}$	...
2	$c_{0,2}$	$c_{1,3}$	$c_{2,4}$	...
$\vdots$	$\vdots$			

Example 3.7. *We use the construction described above to find the minimal graded resolution of the monomial ideal* $I =< x^2, xy, z^2, y^3 >$ *in the polynomial ring* $R = \mathbb{R}[x, y, z]$*. We set up the* $0th$ *and first homological degrees with little computation as follows:*

$$\cdots \longrightarrow R^4 \xrightarrow{\begin{pmatrix} x^2 & xy & z^2 & y^3 \end{pmatrix}} R \xrightarrow{\partial_0} R/I \longrightarrow 0$$

Where $\partial_0 : R \to R/I$ *is the natural projection* $\partial_0(f) = f + I$*. We obtain shiftings on* R^4 *from* $\deg(x^2) = \deg(xy) = \deg(z^2) = 2$*, giving three copies of* $R(-2)$*, and* $\deg(y^3) = 3$*, giving one copy of* $R(-3)$*.* R^4 *is twisted to* $R(-2)^3 \bigoplus R(-3)$ *and that information will be reflected in the Betti table.*

To obtain the next map ∂_2*, find minimal generators of* $\ker \begin{pmatrix} x^2 & xy & y^3 & z^2 \end{pmatrix}$ *We obtain generators of the Kernel by establishing pairwise relations between entries of the matrix representation of* ∂_1 *and make them minimal by reducing the set to a linearly independent set.*
$\ker \begin{pmatrix} x^2 & xy & y^3 & z^2 \end{pmatrix}$ *is generated by*

$$\left\{ \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \\ 0 \\ -x^2 \end{pmatrix}, \begin{pmatrix} z^2 \\ 0 \\ -x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \\ 0 \\ -x \end{pmatrix}, \begin{pmatrix} 0 \\ z^2 \\ -xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -y^3 \\ z^2 \end{pmatrix} \right\}$$

This set does not minimally generate the Kernel because

$$x \begin{pmatrix} 0 \\ y^2 \\ 0 \\ -x \end{pmatrix} + y^2 \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y^3 \\ 0 \\ 0 \\ -x^2 \end{pmatrix}$$

So we can omit the column that would contain $\begin{pmatrix} y^3 \\ 0 \\ 0 \\ -x^2 \end{pmatrix}$ *and get the following presentation matrix of* ∂_2 *with linearly independent columns*

$$\begin{pmatrix} y & z^2 & 0 & 0 & 0 \\ -x & 0 & y^2 & z^2 & 0 \\ 0 & -x^2 & 0 & -xy & -y^3 \\ 0 & 0 & -x & 0 & z^2 \end{pmatrix}$$

Given this presentation matrix, we are able to determine the necessary shiftings of the modules in the second homological degree.

$$R(-3) \bigoplus R(-4) \bigoplus R(-4) \bigoplus R(-4) \bigoplus R(-5)$$
$$= R(-3) \bigoplus R(-4)^3 \bigoplus R(-5)$$

Without considering the shiftings given above, the Kernel of ∂_2 *can be thought of as the set*

$$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \in R^5 \mid \begin{array}{c} ay + bz^2 = 0, \\ -ax + cy^2 + dz^2 = 0, \\ -bx^2 - dxy - ey^3 = 0, \text{ and} \\ -cx + ez^2 = 0 \end{array} \right\}$$

To obtain one generator of the Kernel, consider that when $a = z^2$*, then* $b = -y$ *to satisfy the first equation. Then evaluating the second equation as* $-z^2x + cy^2 + dz^2 = 0$*, an intuitive choice is* $c = 0, d = x$*. Then the fourth equation evaluated as* $0x + ez^2 = 0$ *implies that* $e = 0$*.*

By this process, we obtain $\begin{pmatrix} z^2 \\ -y \\ 0 \\ x \\ 0 \end{pmatrix}$ *as a generator of the kernel.*

Starting with the assumption that $c = z^2$*, follow a similar process to find that* $a = 0, b = 0, d = y^2$*, and* $e = x$ *satisfies the equations above and gives another generator* $\begin{pmatrix} 0 \\ 0 \\ z^2 \\ y^2 \\ x \end{pmatrix}$.

Any other starting assumptions on a or c would be a linear combination of the assumptions we have already considered to satisfy the first and fourth equations given in the representation of the Kernel, so these two linearly independent vectors generate $\ker \partial_2$*.*

We can now give the module of homological degree 3 and the map ∂_3 *by*

$$R^2 \xrightarrow{\begin{pmatrix} z^2 & 0 \\ -y & 0 \\ 0 & z^2 \\ x & y^2 \\ 0 & x \end{pmatrix}} R^5$$

The Kernel of ∂_3 *without considering the shiftings can be thought of as the set*

$$\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in R^2 \mid \begin{array}{c} az^2 = 0 \\ -ya = 0 \\ bz^2 = 0 \\ ax + by^2 = 0, \text{ and} \\ xb = 0 \end{array} \right\}$$

Clearly, the Kernel is generated by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$*, so at homological degree 4 we map 0 to* $R(-5) \bigoplus R(-6)$ *by the zero map.*

One can also deduce that ∂_4 *is the zero map by observing that* ∂_3 *is injective so it's kernel can only be the zero vector.*

The resulting minimal graded free resolution of $I =< x^2, xy, z^2, y^3 >$ *is*

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} z^2 & 0 \\ -y & 0 \\ 0 & z^2 \\ x & y^2 \\ 0 & x \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} y & z^2 & 0 & 0 & 0 \\ -x & 0 & y^2 & z^2 & 0 \\ 0 & -x^2 & 0 & -xy & -y^3 \\ 0 & 0 & -x & 0 & z^2 \end{pmatrix}}$$

$$\longrightarrow R^4 \xrightarrow{\begin{pmatrix} x^2 & xy & z^2 & y^3 \end{pmatrix}} R \xrightarrow{\partial_0} R/I \longrightarrow 0$$

The Betti table of this resolution is

	0	1	2	3
0	1	0	0	0
1	0	3	1	0
2	0	1	3	1
3	0	0	1	1
4	0	0	0	0

As previously described, the presentation matrix of ∂_1 *contains the generators of the ideal and the presentation matrix of* ∂_2 *describes relations between the generators. For example,* $y(x^2) - x(xy) + 0(z^2) + 0(y^3) = 0$ *describes a relation between the generators, so* $\begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}$ *is a generator of* $\ker \begin{pmatrix} x^2 & xy & z^2 & y^3 \end{pmatrix}$ *and appears as a column in the presentation matrix of* ∂_2*.* ∂_3 *representing relations among relations is more abstract, but you can observe that the composition of the two maps is certainly the* 0 *map.*

The following example briefly describes the process of obtaining the minimal graded free resolution of a monomial ideal.

Example 3.8. *In* $R = k[x, y, z]$*, take the ideal* $I =< x^2y, x^2z, xyz >$*. To find the minimal resolution associated to this ideal, generate a linearly independent matrix* A *such that*

$$\begin{pmatrix} x^2y & x^2z & xyz \end{pmatrix} \cdot A = 0.$$

This matrix is given by the linearly independent generators of the kernel of $\begin{pmatrix} x^2y & x^2z & xyz \end{pmatrix}$*. The relations between the generators of the monomial ideal give*

$$\left\{\begin{pmatrix} z \\ -y \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ -x \end{pmatrix}\right\}$$

These vectors are linearly dependent so omit one to obtain the matrix with linearly independent columns

$$\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}$$

This gives the graded minimal resolution

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2y & x^2z & xyz \end{pmatrix}} R \longrightarrow R/I$$

The Betti table associated to this resolution is

	0	*1*	*2*
0	*1*	*0*	*0*
1	*0*	*0*	*0*
2	*0*	*3*	*2*

While the matrix representations for higher differentials become increasingly complicated, computational tools such as Macaulay2 and the complexes package provide an efficient approach for calculating the minimal graded free resolution of monomial ideals. Example 3.7 demonstrates how the minimal graded free resolution can be constructed manually. Once the computation process is well understood, we can confidently rely on software tools to handle more intricate computations and automate the process of obtaining minimal free resolutions of monomial ideals.

In comparing the two distinct resolutions in Example 3.5 and Example 3.7 of the same monomial ideal, we observe an example where the Taylor resolution of a monomial ideal is nonminimal. This prompts the question of what modifications can be made to the Taylor resolution that reduce the ranks of the free modules while ensuring that it remains a valid resolution.

Lyubeznik Construction: [Lyu88]

Lyubeznik gave a construction of a subcomplex of the Taylor resolution which is also a free resolution . In the subcomplex, the module at each homological dimension i by elements $[m_{i_t}, \dots, m_{i_s}]$ in the usual ordering of $F_i(I)$ such that m_j does not divide $\operatorname{lcm}(m_{i_t}, \dots, m_{i_s})$ for all $t < s$ and $j < i_t$.

Example 3.9. *In his 1988 thesis introducing the resolution, Lyubeznik gives an example of an ideal whose Taylor resolution is nonminimal and whose resulting resolution after the truncating process is also nonminimal. The ideal is* $< x_1x_2, x_1x_3, x_2x_3, x_2x_4, x_1x_5, x_3x_6 >$. *By just observing that* $\operatorname{lcm}(x_1x_2, x_1x_3) = \operatorname{lcm}(x_1x_2, x_1x_3, x_2x_3) = \operatorname{lcm}(x_1x_2, x_2x_3)$, *we can tell that its Taylor resolution is nonminimal.*

Example 3.10. *Recall the Taylor resolution found in Example 3.5.*

We follow Lyubeznik's truncation process by eliminating columns of the presentations corresponding to basis elements which do not satisfy the property described by Lyubeznik. $\operatorname{lcm}(x^2, xy, z^2, y^3) = x^2y^3z^2 = \operatorname{lcm}(x^2, z^2, y^3)$, *so the basis element* $\begin{bmatrix} x^2 & xy & z^2 & y^3 \end{bmatrix}$ *of* $R^{F_4(I)}$ *should be omitted. The module in the* $4th$ *homological degree is thus* 0.

Furthermore, $\operatorname{lcm}(x^2, xy, y^3) = \operatorname{lcm}(x^2, y^3)$

$$y^2 \begin{pmatrix} z^2 \\ -y \\ 0 \\ x \\ 0 \\ 0 \end{pmatrix} - z^2 \begin{pmatrix} y^2 \\ 0 \\ -1 \\ 0 \\ x \\ 0 \end{pmatrix} - x \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^2 \\ -z^2 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ y^3 \\ -z^2 \\ 0 \\ 0 \\ x^2 \end{pmatrix}$$

$$y^2 \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 \\ y^2 \\ 0 \\ -x \end{pmatrix} = \begin{pmatrix} y^3 \\ 0 \\ 0 \\ -x^2 \end{pmatrix},$$

To minimize the resolution, we should omit the column which corresponds to the nonmaximal entry of the next map's presentation matrix. Once the necessary column is omitted and the presentation matrix for ∂_3 is injective, its kernel is trivial and can be minimally mapped onto by the zero map.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y^2 \\ -z^2 \\ 1 \\ -x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} z^2 & y^2 & 0 & 0 \\ -y & 0 & y^3 & 0 \\ 0 & -1 & -z^2 & 0 \\ x & 0 & 0 & y^2 \\ 0 & x & 0 & -z^2 \\ 0 & 0 & x^2 & x \end{pmatrix}}$$

$$0 \longrightarrow R^3 \xrightarrow{\begin{pmatrix} z^2 & y^2 & 0 \\ -y & 0 & 0 \\ 0 & -1 & 0 \\ x & 0 & y^2 \\ 0 & x & -z^2 \\ 0 & 0 & x \end{pmatrix}}$$

Now we interpret the nonmaximal entry in the third row of ∂_3 to also represent a linear combination of the columns of ∂_2's presentation matrix.

$$y^2 \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 \\ y^2 \\ 0 \\ -x \end{pmatrix} = \begin{pmatrix} y^3 \\ 0 \\ 0 \\ -x^2 \end{pmatrix}$$

So we can omit the third column from the matrix respresenting ∂_2, making sure to also omit the corresponding copy of R in the second homological degree and omit the corresponding row of the presentation matrix of ∂_3. To preserve exactness, we also omit the column which contained the nonminimal entry.

$$R^{3} \xrightarrow{\begin{pmatrix} z^2 & y^2 & 0 \\ -y & 0 & 0 \\ 0 & -1 & 0 \\ x & 0 & y^2 \\ 0 & x & -z^2 \\ 0 & 0 & x \end{pmatrix}} R^{6} \xrightarrow{\begin{pmatrix} y & z^2 & y^3 & 0 & 0 & 0 \\ -x & 0 & 0 & z^2 & y^2 & 0 \\ 0 & -x^2 & 0 & -xy & 0 & y^3 \\ 0 & 0 & -x^2 & 0 & -x & -z^2 \end{pmatrix}}$$

$$R^{2} \xrightarrow{\begin{pmatrix} z^2 & 0 \\ -y & 0 \\ x & y^2 \\ 0 & -z^2 \\ 0 & x \end{pmatrix}} R^{5} \xrightarrow{\begin{pmatrix} y & z^2 & 0 & 0 & 0 \\ -x & 0 & z^2 & y^2 & 0 \\ 0 & -x^2 & -xy & 0 & y^3 \\ 0 & 0 & 0 & -x & -z^2 \end{pmatrix}}$$

We can further minimize the resolution by observing that the columns of ∂_3 are linearly dependent.

$$-y^2 \begin{pmatrix} z^2 \\ -y \\ x \\ 0 \\ 0 \end{pmatrix} + z^2 \begin{pmatrix} y^2 \\ 0 \\ 0 \\ x \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 \\ 0 \\ y^2 \\ -z^2 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ y^3 \\ 0 \\ 0 \\ x^2 \end{pmatrix}$$

So we can omit the third column from the matrix ∂_3

The The Taylor resolution after this minimization process is

$$0 \longrightarrow R^{2} \xrightarrow{\begin{pmatrix} z^2 & 0 \\ -y & 0 \\ x & y^2 \\ 0 & -z^2 \\ 0 & x \end{pmatrix}}$$

$$R^5 \xrightarrow{\begin{pmatrix} y & z^2 & 0 & 0 & 0 \\ -x & 0 & z^2 & y^2 & 0 \\ 0 & -x^2 & -xy & 0 & y^3 \\ 0 & 0 & 0 & -x & -z^2 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x^2 & xy & z^2 & y^3 \end{pmatrix}} R \longrightarrow R/I$$

	0	1	2	3
0	1	0	0	0
1	0	3	1	0
2	0	1	3	1
3	0	0	1	1
4	0	0	0	0

Another notable construction of a free resolution is the Barile-Macchia Resolution. This construction builds the resolution through discrete Morse theory choosing the generators through an algorithmic process using subsets of the generators of the ideal rather than starting with a larger resolution and truncating [BM20].

As seen at the end of Example 3.5, the Taylor resolution of a monomial ideal can be nonminimal and often is. We aim to identify characteristics of the generators of the monomial ideal which imply that the Taylor resolution is nonminimal and characteristics which imply minimality.

A resolution given by matrix presentations, as our Taylor resolutions are, is called nonminimal if some entry of a matrix is not in the maximal ideal $< x_1, \dots, x_n >$ of $F[x_1, \dots, x_n]$. An entry of a matrix not in the maximal ideal of the polynomial ring is a scalar entry, usually 1 or -1. The case of a scalar entry in a matrix occurs when there exists $\{m_j\}_{j\in J} \subseteq \{m_1, \dots, m_r\}$ such that for some $m_k \in \{m_j\}_{j\in J}$,

$$\text{lcm}(m_j, \dots, m_{|J|}) = \text{lcm}(m_j, \dots, \hat{m_k}, \dots, m_{|J|}), \text{ so } \frac{\text{lcm}(m_j, \dots, m_{|J|})}{\text{lcm}(m_j, \dots, \hat{m_k}, \dots, m_{|J|})} = 1$$

When an entry of 1 or -1 occurs in the jth row of the ith presentation matrix, then the jth column of of the $(i-1)$th presentation matrix is a linear combination of the other columns corresponding to the other nonzero entries in the jth row of the ith presentation matrix. Recall that the construction of the minimal graded free resolution required that the generators of the kernel must be minimal.

There is one minimal free resolution associated to a monomial ideal, so the ranks of the modules in the resolution are the smallest possible for that ideal.

Note that

$$F_0(I) = \{\varnothing\},$$

$$F_1(I) = \{[m_1], \dots, [m_r]\},$$

$$F_{r-1}(I) = \{[m_2, \dots, m_r], \dots, [m_1, \dots, \hat{m}_i, \dots, m_r] \dots, [m_1, \dots, m_{r-1}]\}$$

where $\hat{m}_i$ indicates to omit m_i, and

$$F_r(I) = \{[m_1, \dots, m_r]\}.$$

The cardinality of each of these sets is $\binom{r}{i}$, but that is usually larger than the ith rank in the minimal resolution.

In taking the least common multiples of these sets of monomial generators, there are fewer than $\binom{r}{i}$ unique lcms if and only if $\mathrm{lcm}(m_j)_{j\in J} = \mathrm{lcm}(m_k)_{k\in K}$ for some distinct bijective subsets $\{m_j\}_{j\in J}, \{m_k\}_{k\in K} \subset \{m_1, \dots, m_r\}$.

This case entails that truncation is possible in the Taylor resolution and it was not minimal to begin with.

Example 3.11. *In the Examples 3.5 and 3.7, we found two different resolutions asociated to one ideal. The discrepancy between the two resolutions can be attributed to the fact that the Taylor resolution of that ideal is nonminimal. In the case that Taylor's resolution is minimal, it will coincide with the minimal free resolution. Take the ideal $I =< xy, y^2, z^2 >$ and note that each pairwise least common multiple of the monomial generators is distinct. Here is the resolution to the 2nd homological degree*

$$R^3 \xrightarrow{\begin{pmatrix} xy & y^2 & z^2 \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Establish pairwise relations between the generators and check for linear dependence to obtain generators for ∂_2.

$$\left\{ \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \begin{pmatrix} z^2 \\ 0 \\ -xy \end{pmatrix}, \begin{pmatrix} 0 \\ z^2 \\ -y^2 \end{pmatrix} \right\}$$

The set is linearly independent so we cannot omit any column of the matrix created from these vectors.

$$R^3 \xrightarrow{\begin{pmatrix} y & z^2 & 0 \\ -x & 0 & z^2 \\ 0 & -xy & -y^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} xy & y^2 & z^2 \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Express the Kernel of ∂_2 *as*

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in R^3 \mid \begin{array}{r} ay + bz^2 = 0, \\ -ax + cz^2 = 0, \\ -bxy - cy^2 = 0 \end{array} \right\}$$

The kernel is generated by $\begin{pmatrix} z^2 \\ -y \\ x \end{pmatrix}$, *so the final minimal graded resolution is*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} z^2 \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y & z^2 & 0 \\ -x & 0 & z^2 \\ 0 & -xy & -y^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} xy & y^2 & z^2 \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

	0	*1*	*2*	*3*
0	*1*	*0*	*0*	*0*
1	*0*	*3*	*1*	*0*
2	*0*	*0*	*2*	*0*
3	*0*	*0*	*0*	*1*
4	*0*	*0*	*0*	*0*

We can see that the minimal resolution coincides with the Taylor resolution because the module rank at the ith homological degree in the resolution is $\binom{3}{i}$, the same as it would be in Taylor's resolution.

4. Minimal Resolutions in Bivariate Polynomial Rings

In the cases of polynomial rings in the fewest variables, the possible resolutions follow some forms due because there are few possible relations to have between minimal generators in one or two variables. As explored in Example 2.12, a generator of an ideal is condidered redundant if it is a multiple of another generator. As such, a monomial ideal in $R = F[x]$ can only have one generator and will have the form $< x^j >$ for some $j \in \mathbb{N}$. This monomial ideal will have the following minimal graded free resolution:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x^j \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

To figure out a form for monomial ideals in two variables, first consider the cases of generators of low degree:

When $I =< x, y >$,

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

When $I =< x^2, y >$,

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x^2 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^2 & y \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

When $I =< x^2, xy >$,

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

The only way to minimally generate a monomial ideal in two variables is by the form $< x^{a_1}y^{b_1}, ..., x^{a_n}y^{b_n} >$ such that for any i, j distinct in $1, ..., n$ then either $a_i < a_j$ and $b_j < b_i$ or $a_j < a_i$ and $b_i < b_j$. If $a_i < a_j$ and $b_i < b_j$ then $x^{a_j}y^{b_j}$ is a multiple of $x^{a_i}y^{b_i}$.

We can use these conditions on the generators to give a general for the formula of a minimal graded free resolution of a monomial ideal in two variables by the number of generators.

Take an ideal with three monomial generators $x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}$ such that $\min(a_1, a_2, a_3) = a_1$ and $\max(b_1, b_2, b_3) = b_1$. This ordering is given to satisfy the conditions on the exponents given above and to ensure that the differences we take in these exponents are positive.

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x^{a_2-a_1} & x^{a_3-a_1} \\ -y^{b_1-b_2} & 0 \\ 0 & -y^{b_1-b_3} \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^{a_1}y^{b_1} & x^{a_2}y^{b_2} & x^{a_3}y^{b_3} \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

In n generators ordered by $<$, $I =< x^{a_1}y^{b_1}, ..., x^{a_n}y^{b_n} >$ and the minimal resolution has the form

$$0 \longrightarrow R^{n-1} \xrightarrow{\begin{pmatrix} x^{a_2-a_1} & 0 & 0 & \cdots & 0 & 0 \\ -y^{b_1-b_2} & x^{a_3-a_2} & 0 & \cdots & 0 & 0 \\ 0 & -y^{b_2-b_3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & x^{a_4-a_3} & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x^{a_{n-1}-a_{n-2}} & 0 \\ 0 & 0 & 0 & \cdots & -y^{b_{n-2}-b_{n-1}} & x^{a_n-a_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & -y^{b_{n-1}-b_n} \end{pmatrix}}$$

$$R^n \xrightarrow{\begin{pmatrix} x^{a_1}y^{b_1} & \cdots & x^{a_n}y^{b_n} \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Example 4.1. *The following image is the minimal free resolution of* $I =< x^{15}y^5, x^{10}y^{10}, x^{12}y^7, x^7y^{12} >$ *was computed using Macaulay2 and adheres the form given above. Note that Macaulay2 gives resolutions with arrows pointing left instead of arrows pointing right as this thesis has done. Macaulay2 also omits* $R/I \to 0$.

```
i17 : resolution monomialIdeal(x^15*y^5, x^10*y^10, x^12*y^7, x^7*y^12)
```

$$\text{o17} = \underset{0}{R^1} \xleftarrow{\begin{pmatrix} x^{12}y^7 & x^7y^{12} & x^{15}y^5 & x^{10}y^{10} \end{pmatrix}} \underset{1}{R^4} \xleftarrow{\begin{pmatrix} -x^3 & -y^3 & 0 \\ 0 & 0 & -x^3 \\ y^2 & 0 & 0 \\ 0 & x^2 & y^2 \end{pmatrix}} \underset{2}{R^3} \xleftarrow{0} \underset{3}{0}$$

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