An exploration of alternative approaches to the representation of uncertainty in model predictions

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Abstract

Several simple test problems are used to explore the following approaches to the representation of the uncertainty in model predictions that derives from uncertainty in model inputs: probability theory, evidence theory, possibility theory, and interval analysis. Each of the test problems has rather diffuse characterizations of the uncertainty in model inputs obtained from one or more equally credible sources. These given uncertainty characterizations are translated into the mathematical structure associated with each of the indicated approaches to the representation of uncertainty and then propagated through the model with Monte Carlo techniques to obtain the corresponding representation of the uncertainty in one or more model predictions. The different approaches to the representation of uncertainty can lead to very different appearing representations of the uncertainty in model predictions even though the starting information is exactly the same for each approach. To avoid misunderstandings and, potentially, bad decisions, these representations must be interpreted in the context of the theory/procedure from which they derive.

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1. Introduction

The need for an appropriate representation of uncertainty as part of any analysis that supports an important decision is now almost universally recognized \cite{1–9}. Traditionally, probability theory has provided the language and mathematics for the representation of uncertainty \cite{10–17}. More recently, other languages and mathematical structures for the representation of uncertainty have been introduced, including evidence theory \cite{18–24}, possibility theory \cite{25–29}, and interval analysis \cite{30–35}. A number of comparative discussions of different approaches to the representation of uncertainty are available \cite{36–41}.

The introduction of these newer representational structures for uncertainty has been accompanied by a lively discussion and debate of their various merits and demerits \cite{42–50} (see Ref. \cite{41}, Chapter 1, for additional discussion). At the same time, steadily increasing computational power has made the analysis of uncertainty increasingly practicable, and the increasing use of simulation in support of decision making has created a demand for informative, decision-aiding characterizations of the uncertainty in analysis outcomes relevant to these decisions. As a result, there is wide spread interest in both the interpretation and the computational implementation of these various representations for uncertainty.

The purpose of this paper is to explore and compare the use of several uncertainty representations (i.e. probability theory, evidence theory, possibility theory, and interval analysis) on a suite of simple test problems proposed as part of a workshop on the representation of uncertainty \cite{51}. The test problems involve various combinations of epistemic and aleatory uncertainty, with the designator epistemic being used in reference to an uncertainty that derives from a lack of knowledge with respect to a fixed but poorly known parameter or other entity used in an analysis and the designator aleatory being used in reference to an uncertainty that derives from an inherent variability or randomness associated with the behavior of a system \cite{9}. The problems themselves are very simple. Thus, the object of the paper is not to solve a real problem but rather to use these simple problems to illustrate the indicated uncertainty
representations and considerations associated with their computational implementation.

The paper is organized as follows. Initially, overview descriptions are presented for probability theory (Section 2), evidence theory (Section 3), possibility theory (Section 4), and interval analysis (Section 5). These descriptions include the introduction of Monte Carlo techniques that will be used to obtain numerical results for the problems under consideration. Several different formulations (i.e. initial specifications) of the uncertainty in the parameters used in a simple algebraic model are considered in Section 6. All formulations in Section 6 involve only epistemic uncertainty, are incorporated into uncertainty representations based on probability theory, evidence theory, and possibility theory, respectively, and are propagated through the model with Monte Carlo techniques to obtain corresponding representations of uncertainty in model predictions. Further, here and in other problems, the Monte Carlo propagation of uncertainty also provides an interval analysis representation of the uncertainty in model predictions. A more complex problem involving only epistemic uncertainty is introduced in Section 7 and evaluated with techniques based on probability theory, evidence theory, and possibility theory. The conceptual complexity of the problems under consideration is then extended in Section 8 by the introduction of methods that involve both aleatory and epistemic uncertainty. In these problems, aleatory uncertainty is represented with probability theory, and representations of epistemic uncertainty with probability theory, evidence theory, and possibility theory are presented. Then, a problem involving both aleatory and epistemic uncertainty for a simple mass-spring-damper system is presented (Section 9). Finally, the presentation ends with a concluding discussion (Section 10).

2. Probability theory

Probability theory provides the mathematical structure traditionally used to represent uncertainty and is based on assigning probabilities to subsets of a universal set \( \mathcal{S} \). The assigned probabilities represent the amount of ‘likelihood’ associated with various subsets of \( \mathcal{S} \). Formally, probability is defined by a triple \((\mathcal{S}, \mathcal{F}, P)\) called a probability space, where (i) \( \mathcal{S} \) is a set that contains everything that could occur in the particular universe under consideration, (ii) \( \mathcal{F} \) is a suitably restricted set of subsets of \( \mathcal{S} \), and (iii) \( P \) is the function that defines probability for elements of \( \mathcal{S} \) (Ref. [52], Section IV.4).

The set \( \mathcal{F} \) is required to have the properties that (i) if \( \delta \in \mathcal{S} \), then \( \delta^c \in \mathcal{S} \), where \( \delta^c \) is the complement of \( \delta \), and (ii) if \( \delta_1, \delta_2, \ldots \) is a sequence of elements of \( \mathcal{S} \), then \( \bigcup_i \delta_i \in \mathcal{S} \) and \( \bigcap_i \delta_i \in \mathcal{S} \). Further, \( P \) is required to have the properties that (i) if \( \delta \in \mathcal{S} \), then \( 0 \leq p(\delta) \leq 1 \), (ii) \( p(\mathcal{S}) = 1 \), and (iii) if \( \delta_1, \delta_2, \ldots \) is a sequence of disjoint sets from \( \mathcal{S} \), then \( p(\bigcup_i \delta_i) = \sum_i p(\delta_i) \) (Ref. [52], Section IV.3). In the terminology of probability theory, \( \mathcal{S} \) is called the sample space or universal set; elements of \( \mathcal{S} \) are called elementary events; subsets of \( \mathcal{S} \) contained in \( \mathcal{S} \) are called events; the set \( \mathcal{S} \) itself has the properties of what is called a \( \sigma \)-algebra (Ref. [52], Section IV.3); and \( p \) is called a probability measure.

One of the important properties of probability is that \( p(\delta) + p(\delta^c) = 1 \) \( (2.1) \) for \( \delta \in \mathcal{S} \). In words, the probability of an event occurring (i.e. \( p(\delta) \)) and the probability of an event not occurring (i.e. \( p(\delta^c) \)) must sum to one. Thus, specification of the likelihood of an event occurring in probability theory also results in, or implies, a specification of the likelihood of that event not occurring. As discussed in Sections 3 and 4, less restrictive conditions on the specification of likelihood are present in evidence theory and possibility theory.

The sample spaces considered in this presentation will contain either real numbers or vectors of real numbers. When the sample space \( \mathcal{S} \) contains real numbers, cumulative distribution functions (CDFs) and complementary cumulative distribution functions (CCDFs) provide visual summaries of the information contained in the probability space \((\mathcal{S}, \mathcal{F}, P)\). Specifically, CDFs and CCDFs are defined by the sets of points

\[
\mathcal{CDF} = \{[v, \mu(\mathcal{S})], v \in \mathcal{S}\}
\]

and

\[
\mathcal{CCDF} = \{[v, \mu(\mathcal{S})], v \in \mathcal{S}\},
\]

respectively, where

\[
\mathcal{S} = \{x : x \in \mathcal{S} \text{ and } x > v\}.
\]

In the preceding, \( \mu(\mathcal{S}) \) is the probability that a value smaller than or equal to \( v \) will occur, and \( \mu(\mathcal{S}) \) is the probability that a value larger than \( v \) will occur. Thus, a plot of the points in \( \mathcal{CDF} \) provides a visual representation of the probability of having values less than or equal to individual elements of \( \mathcal{S} \), and a plot of the points in \( \mathcal{CCDF} \) provides a visual representation of the probability of having values larger than individual elements of \( \mathcal{S} \) (Fig. 1).

Complementary cumulative distribution functions are widely used to display the results of risk assessments for two reasons. First, CCDFs answer the question how likely is an event to be this large or larger, which is typically the large consequences. As discussed in Sections 3 and 4, less restrictive conditions on the specification of likelihood are present in evidence theory and possibility theory.

The primary focus in many, if not most, problems involving probability is on functions

\[
y = f(x)
\]

defined for elements \( x \) of some sample space \( \mathcal{X} \) that is part of a probability space \((\mathcal{X}, \mathcal{G}, P)\). In practice, \( f \) can be quite complex (e.g. the numerical solution of a system of nonlinear partial differential equations), and the dimensionality of the vector \( x \) of uncertain analysis inputs can be high. The analysis outcome \( y \) is also typically a vector \( y \) of high
dimensionality but is indicated in Eq. (2.5) as being real-valued for notational convenience.

The sample space \( \mathcal{X} \) constitutes the domain for the function \( f \) in Eq. (2.5). In turn, the range of \( f \) is given by the set

\[
\mathcal{Y} = \{ y : y = f(\mathbf{x}), \mathbf{x} \in \mathcal{X} \}.
\]  

(2.6)

The uncertainty in the values of \( y \) contained in \( \mathcal{Y} \) derives from the probability space \( (\mathcal{X}, \mathcal{F}, p_Y) \) that characterizes the uncertainty in \( x \) and from the properties of the function \( f \). In concept, \( (\mathcal{X}, \mathcal{F}, p_X) \) and \( f \) induce a probability space \( (\mathcal{Y}, \mathcal{G}, p_Y) \). The probability \( p_Y \) is defined for a subset \( \delta \) of \( \mathcal{Y} \) by

\[
p_Y(\delta) = p_X(f^{-1}(\delta)),
\]  

(2.7)

where

\[
f^{-1}(\delta) = \{ \mathbf{x} : \mathbf{x} \in \mathcal{X} \text{ and } y = f(\mathbf{x}) \in \delta \}.
\]  

(2.8)

A formal development of \( (\mathcal{Y}, \mathcal{G}, p_Y) \) would focus on the properties that \( f \) must satisfy to actually produce this probability space (Ref. [52], Section IV.4; Ref. [53], Sections 4.6 and 4.7); such details are outside the scope of this presentation.

As discussed in conjunction with Eqs. (2.2)–(2.4), the uncertainty in \( y \) characterized by \( (\mathcal{Y}, \mathcal{G}, p_Y) \) is typically summarized with CDFs and CCDFs. Such CDFs and CCDFs are defined in a manner analogous to that shown in Eqs. (2.2) and (2.3), with

\[
\mathcal{C}_Y \mathcal{F} = \{ [v, p_Y(\mathcal{G}_v)], v \in \mathcal{Y} \} = \{ [v, p_X(f^{-1}(\mathcal{G}_v))], v \in \mathcal{Y} \},
\]  

(2.9)

\[
\mathcal{C}_Y \mathcal{C}_Y \mathcal{F} = \{ [v, p_Y(\mathcal{G}_v)], v \in \mathcal{Y} \} = \{ [v, p_X(f^{-1}(\mathcal{G}_v))], v \in \mathcal{Y} \},
\]  

(2.10)

\[
\mathcal{G}_v = \{ y : y \in \mathcal{Y} \text{ and } y > v \}.
\]  

(2.11)

Plots of the points contained in \( \mathcal{C}_Y \mathcal{F} \) and \( \mathcal{C}_Y \mathcal{C}_Y \mathcal{F} \) produce a figure similar to Fig. 1 and provide a visual representation of the uncertainty in \( y \).

The definition of \( p_Y(\delta) \) in Eq. (2.7) as the probability \( p_X(f^{-1}(\delta)) \) of \( f^{-1}(\delta) \) is presented to facilitate comparisons at a later point in this presentation between the use of probability theory, evidence theory, and possibility theory in the representation of the uncertainty in model predictions (i.e. in function evaluations of form indicated in Eq. (2.5)). In practice, \( p_Y(\delta) \) is not found by determining \( f^{-1}(\delta) \) and then evaluating \( p_X(\delta) \). Rather, a Monte Carlo procedure is used to estimate \( p_Y(\delta) \). Such a procedure is based on the relationship

\[
p_Y(\delta) = \int_{\delta} \delta_Y[f(x)]dx(x)dV = \frac{1}{nS} \sum_{i=1}^{nS} \delta_Y[f(x_i)],
\]  

(2.12)

where \( \delta_Y[f(x)] = 1 \) if \( f(x) \in \delta \) and 0 otherwise, \( dV \) is the density function associated with \( (\mathcal{X}, \mathcal{F}, p_X) \) (i.e. \( dV \) is a function with the property that \( p_X(\mathcal{G}) = \int_{\mathcal{G}} dV \) for \( \mathcal{G} \in \mathcal{X} \)).

As indicated in conjunction with Eq. (2.5), the elements \( \mathbf{x} \) of the sample space \( \mathcal{X} \) associated with a probability space \( (\mathcal{X}, \mathcal{F}, p_X) \) are often vectors. Specifically, \( \mathbf{x} \) has the form

\[
\mathbf{x} = [x_1, x_2, \ldots, x_n],
\]  

(2.13)

where the uncertainty in each \( x_i \) is characterized by a probability space \( (\mathcal{X}_i, \mathcal{F}_i, p_{X_i}) \). Most problems start with the probability spaces \( (\mathcal{X}_i, \mathcal{F}_i, p_{X_i}), i = 1, 2, \ldots, n \), and then construct the probability space \( (\mathcal{X}, \mathcal{F}, p_X) \) for \( \mathbf{x} \). In this construction,

\[
\mathcal{X} = \{ \mathbf{x} : \mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \}
\]  

(2.14)

and \( \mathcal{X} \) is developed from the sets contained in

\[
\mathcal{C} = \{ \delta : \delta = \delta_1 \times \delta_2 \times \cdots \times \delta_n \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \}
\]  

(2.15)

(see Ref. [52], Section IV.6 and Ref. [53], Section 2.6). Further, if the \( x_i \) are independent (i.e. if the occurrence of one \( x_i \) has no implications for the occurrence of the remaining \( x_j, j \neq i \)), then

\[
p_X(\delta) = \prod_{i=1}^{n} p_i(\delta_i)
\]  

(2.16)

for \( \delta = \delta_1 \times \delta_2 \times \cdots \times \delta_n \in \mathcal{C} \) and, more generally,

\[
p_X(\delta) = \int_{\delta} dV(x)dV
\]  

(2.17)
for $\delta \in \mathcal{X}$, where
\[ d_x(x) = \prod_{i=1}^{n} d_i(x_i) \tag{2.18} \]
is the density function associated with $(\mathcal{A}, \mathcal{X}, \mathcal{A}_x)$ and $d_i$ is the density function associated with $(\mathcal{A}_i, \mathcal{X}_i, \mathcal{A}_{x_i})$ for $i = 1, 2, \ldots, n$. The definition of $\mathcal{A}_x$ and $d_x$ are more complex when the $x_i$ are not independent and will not be considered here.

Additional information on probability theory can be found in a large number of excellent texts [52–56].

3. Evidence theory

Evidence theory [18–24] provides an alternative to the traditional manner in which probability theory is used to represent uncertainty by allowing less restrictive statements about likelihood than is the case with a full probabilistic specification of uncertainty. In particular, evidence theory involves two specifications of likelihood, a belief and a plausibility, for each subset of the universal set under consideration.

Formally, an application of evidence theory involves the specification of a triple $(\mathcal{A}, \mathcal{S}, m)$, where (i) $\mathcal{A}$ is a set that contains everything that could occur in the particular universe under consideration, (ii) $\mathcal{S}$ is a countable collection of subsets of $\mathcal{A}$, and (iii) $m$ is a function defined on subsets of $\mathcal{A}$ such that
\[ m(\emptyset) = \begin{cases} 0 & \text{if } \emptyset \in \mathcal{S} \\ 0 & \text{if } \emptyset \subset \mathcal{A} \text{ and } \emptyset \notin \mathcal{S} \end{cases} \tag{3.1} \]
and
\[ \sum_{\delta \in \mathcal{S}} m(\delta) = 1. \tag{3.2} \]
For a subset $\delta$ of $\mathcal{A}$, $m(\delta)$ is a number characterizing the amount of likelihood that can be assigned to $\delta$ but to no proper subset of $\delta$. In the terminology of evidence theory, (i) $\mathcal{A}$ is the sample space or universal set, (ii) $\mathcal{S}$ is the set of focal elements for $\mathcal{A}$ and $m$, and (iii) $m(\delta)$ is the basic probability assignment (BPA) associated with a subset $\delta$ of $\mathcal{A}$.

The sample space $\mathcal{A}$ plays the same role in both probability theory and evidence theory. However, the set $\mathcal{S}$ has a different character in the two theories. In probability theory, $\mathcal{S}$ has special algebraic properties fundamental to the development of probability (see Section 2) and contains all subsets of $\mathcal{A}$ for which probability is defined. In evidence theory, $\mathcal{S}$ has no special algebraic properties (i.e. $\mathcal{S}$ is not required to be a $\sigma$-algebra as is the case in probability theory) and contains the subsets of $\mathcal{A}$ with nonzero BPs.

In probability theory, the function $p$ actually defines the probabilities for elements of $\mathcal{S}$, with these probabilities being the fundamental measure of likelihood. In evidence theory, the function $m$ is not the fundamental measure of likelihood. Rather, there are two measures of likelihood, called belief and plausibility, that are obtained from $m$ as described in the next paragraph. The designation BPA for $m(\delta)$ is almost universally used, but unfortunately, as $m$ does not define probabilities except under very special circumstances. Given the requirement in Eq. (3.2), the set $\mathcal{S}$ of focal elements associated with an evidence space $(\mathcal{A}, \mathcal{S}, m)$ can contain at most a countable number of elements; in contrast, the set $\mathcal{S}$ of events associated with a probability space $(\mathcal{A}, \mathcal{S}, p)$ can, and usually does in most problems, contain an uncountably infinite number of elements.

The belief, $\text{Bel}(\delta)$, and plausibility, $\text{Pl}(\delta)$, for a subset $\delta$ of $\mathcal{A}$ are defined by
\[ \text{Bel}(\delta) = \sum_{\emptyset \subset \delta} m(\emptyset) \tag{3.3} \]
and
\[ \text{Pl}(\delta) = \sum_{\emptyset \cap \delta \neq \emptyset} m(\emptyset). \tag{3.4} \]
In concept, $m(\emptyset)$ can be thought of as the amount of likelihood that is associated with $\emptyset$ but without any specification of how this likelihood might be apportioned over $\emptyset$. Thus, the likelihood might be associated with any subset of $\mathcal{A}$. Given the preceding conceptualization of $m(\emptyset)$, the belief $\text{Bel}(\delta)$ can be viewed as the minimum amount of likelihood that must be associated with $\delta$ (i.e. this amount of likelihood cannot move out of $\delta$ because the summation in Eq. (3.3) only involves $\emptyset$ that satisfy $\emptyset \subset \delta$). Similarly, the plausibility $\text{Pl}(\delta)$ can be viewed as the maximum amount of likelihood that could be associated with $\delta$ (i.e. this amount of likelihood could move into $\delta$ because the summation in Eq. (3.4) involves all $\emptyset$ that satisfy $\emptyset \cap \delta \neq \emptyset$).

Belief and plausibility satisfy the equality
\[ \text{Bel}(\delta) + \text{Pl}(\delta^c) = 1 \tag{3.5} \]
for every subset $\delta$ of $\mathcal{A}$. In words, the belief in the occurrence of an event (i.e. $\text{Bel}(\delta)$) and the plausibility of the nonoccurrence of an event (i.e. $\text{Pl}(\delta^c)$) must sum to one. Further,
\[ \text{Bel}(\delta) + \text{Bel}(\delta^c) \leq 1 \tag{3.6} \]
and
\[ \text{Pl}(\delta) + \text{Pl}(\delta^c) \geq 1. \tag{3.7} \]
Thus, the specification of belief is capable of incorporating a lack of assurance that is manifested in the sum of the beliefs in the occurrence (i.e. $\text{Bel}(\delta)$) and nonoccurrence (i.e. $\text{Bel}(\delta^c)$) of an event $\delta$ being less than one. Similarly, the specification of plausibility is capable of incorporating a recognition of alternatives that is manifested in the sum of the plausibilities in the occurrence (i.e. $\text{Pl}(\delta)$) and nonoccurrence (i.e. $\text{Pl}(\delta^c)$) of an event $\delta$ being greater
than one. In contrast, probability theory imposes more restrictive conditions on the specification of likelihood as a result of the requirement that the probabilities of the occurrence and nonoccurrence of an event must sum to one (see Eq. (2.1)).

As indicated in conjunction with Eqs. (2.2) and (2.3), CDFs and CCDFs can be used to provide summaries of the information contained in a probability space \((\mathcal{F}, \mathcal{S}, p)\) when \(\mathcal{F}\) contains real numbers (i.e. scalars). Similarly, cumulative belief functions (CBFs), complementary cumulative belief functions (CCBFs), cumulative plausibility functions (CPFps), and complementary cumulative plausibility functions (CCPFs) can be used to summarize beliefs and plausibilities when \(\mathcal{F}\) contains real numbers. Specifically, CBFs, CCBFs, CPFs and CCPFs are defined by the sets of points

\begin{align}
\mathcal{CBF} &= \{v, \text{Bel}(\mathcal{F}_v), v \in \mathcal{F}\} \\
\mathcal{CCBF} &= \{v, \text{Bel}(\mathcal{F}_v), v \in \mathcal{F}\} \\
\mathcal{CPF} &= \{v, \text{Pl}(\mathcal{F}_v), v \in \mathcal{F}\} \\
\mathcal{CCPF} &= \{v, \text{Pl}(\mathcal{F}_v), v \in \mathcal{F}\},
\end{align}

where \(\mathcal{F}_v\) is defined in Eq. (2.4). Plots of the points in the preceding sets produce CBFs, CCBFs, CPFs and CCPFs, respectively (Fig. 2).

As grouped in Fig. 2, a CBF and the corresponding CPF occur together naturally as a pair because, for a given value \(v\) on the abscissa, (i) the value of the CBF (i.e. Bel(\(\mathcal{F}_v\))) is the smallest probability value for \(\mathcal{F}_v\) that is consistent with the information characterized by \((\mathcal{F}, \mathcal{S}, p)\) and (ii) the value of the CPF (i.e. Pl(\(\mathcal{F}_v\))) is the largest probability value for \(\mathcal{F}_v\) that is consistent with the information characterized by \((\mathcal{F}, \mathcal{S}, m)\). A similar interpretation holds for the CCBF and CCPF. Indeed, this bounding relationship occurs for any subset \(\mathcal{E}\) of \(\mathcal{F}\), and thus Pl(\(\mathcal{E}\)) and Bel(\(\mathcal{E}\)) can be thought of as defining upper and lower probabilities for \(\mathcal{E}\) [57–59].

As discussed in conjunction with Eq. (2.5), probability can be used to represent the uncertainty in function evaluations (i.e. system outcomes \(y\)) when the uncertainty associated with the domain of the function is characterized by a probability space \((\mathcal{X}, \mathcal{X}, p_{xy})\). Similarly, belief and plausibility can be used to represent the uncertainty in function evaluations when the uncertainty associated with the domain of the function is characterized by an evidence space \((\mathcal{X}, \mathcal{X}, m_{xy})\).

For the function \(f\) in Eq. (2.5) and an evidence space \((\mathcal{X}, \mathcal{X}, m_{xy})\), the range of \(f\) is given by the set \(\mathcal{Y}\) defined in Eq. (2.6). Analogously to the case for probability, the uncertainty in the values of \(y\) contained in \(\mathcal{Y}\) derives from the evidence space \((\mathcal{X}, \mathcal{X}, m_{xy})\) that characterizes the uncertainty in \(x\) and from the properties of the function \(f\). In concept, \((\mathcal{X}, \mathcal{X}, m_{xy})\) and \(f\) induce an evidence space \((\mathcal{Y}, \mathcal{Y}, m_{yx})\). In practice, \(m_{yx}\) is not determined. Rather, the belief Bel(\(\mathcal{E}\)) and Pl(\(\mathcal{E}\)) for a subset \(\mathcal{E}\) of \(\mathcal{Y}\) are determined from the BPA \(m_{x}\) associated with \((\mathcal{X}, \mathcal{X}, m_{xy})\). In particular,

\begin{align}
\text{Bel}_y(\mathcal{E}) &= \text{Bel}_x(f^{-1}(\mathcal{E})) = \sum_{\mathcal{U} \subseteq f^{-1}(\mathcal{E})} m_x(\mathcal{U}) \\
\text{Pl}_y(\mathcal{E}) &= \text{Pl}_x(f^{-1}(\mathcal{E})) = \sum_{\mathcal{U} \cap f^{-1}(\mathcal{E}) \neq \emptyset} m_x(\mathcal{U}),
\end{align}

where \(f^{-1}(\mathcal{E})\) is defined in Eq. (2.8) and Bel\(_x\) and Pl\(_x\) represent belief and plausibility defined for the evidence space \((\mathcal{X}, \mathcal{X}, m_{xy})\). The preceding definitions for Bel\(_y\) and Pl\(_y\) are analogous to the definition of \(p_y(\mathcal{E})\) in Eq. (2.7).

As discussed in conjunction with Eqs. (2.9)–(2.11), the uncertainty in \(y\) characterized by the evidence space \((\mathcal{Y}, \mathcal{Y}, m_{yx})\) can be summarized with CBFs, CCBFs, CPFs and CCPFs. In particular, CBFs, CCBFs, CPFs and CCPFs

![Frame 2a](image1.png) ![Frame 2b](image2.png)

Fig. 2. Cumulative belief function (CBF), complementary cumulative belief function (CCBF), cumulative plausibility function (CPF), and complementary cumulative plausibility function (CCPF) for a variable \(v\) with values from the interval \([1, 10]\) and each of the following intervals assigned to a BPA of 0.1: \([1, 3]\), \([1, 4]\), \([1, 10]\), \([2, 4]\), \([2, 6]\), \([5, 8]\), \([5, 10]\), \([7, 8]\), \([7, 10]\), \([9, 10]\).
for \( y \) are defined by the sets of points
\[ CBF = \{[v, \text{Bel}_v(\mathcal{Y})], v \in \mathcal{Y} \} \]
\[ CCBF = \{[v, \text{Bel}_v(\mathcal{Y})], v \in \mathcal{Y} \} \] (3.14)
\[ CCPF = \{[v, \text{Pl}_v(\mathcal{Y}^c)], v \in \mathcal{Y} \} \]
\[ CCBF = \{[v, \text{Pl}_v(\mathcal{Y})], v \in \mathcal{Y} \} \] (3.15)
where \( \mathcal{Y} \) is defined in Eq. (2.11). Plots of the points contained in \( CBF, CCBF, CCPF \) and \( CCBF \) produce a figure similar to Fig. 2 and provide a visual representation of the uncertainty in \( y \) in terms of belief and plausibility.

The beliefs and plausibilities appearing in Eqs. (3.14)–(3.17) are defined by sums of BPs for elements of \( \mathcal{X} \). A formal representation for these sums follows. The representation is not necessary at this point in the presentation but will be useful in Section 6 when the construction of CBFs, CCBFs, CPFs and CCPFs is illustrated. For notational convenience, let \( \delta_j \) denote the \( j \)th element of \( \mathcal{X} \) for the evidence space \( \mathcal{X}, \mathcal{X}, m_X \). Such a numbering is possible because \( \mathcal{X} \) is countable; otherwise, the constraint in Eq. (3.2) could not hold. For \( v \in \mathcal{Y} \), let
\[ \mathcal{ICBF}_v = \{j : \delta_j \subset f^{-1}(\mathcal{Y}^c)\} \] (3.18)
\[ \mathcal{ICCBF}_v = \{j : \delta_j \subset f^{-1}(\mathcal{Y})\} \] (3.19)
\[ \mathcal{ICPF}_v = \{j : \delta_j \cap f^{-1}(\mathcal{Y}^c) \neq \emptyset\} \] (3.20)
\[ \mathcal{ICCPF}_v = \{j : \delta_j \cap f^{-1}(\mathcal{Y}) \neq \emptyset\} \] (3.21)
In turn, the beliefs and plausibilities in Eqs. (3.14)–(3.17) are defined by
\[ \text{Bel}_v(\mathcal{Y}^c) = \text{Bel}_v(\mathcal{Y}) \sum_{j \in \mathcal{ICBF}_v} m_X(\delta_j) \] (3.22)
\[ \text{Bel}_v(\mathcal{Y}) = \text{Bel}_v(\mathcal{Y}^c) \sum_{j \in \mathcal{ICCBF}_v} m_X(\delta_j) \] (3.23)
\[ \text{Pl}_v(\mathcal{Y}^c) = \text{Pl}_v(\mathcal{Y}) \sum_{j \in \mathcal{ICPF}_v} m_X(\delta_j) \] (3.24)
\[ \text{Pl}_v(\mathcal{Y}) = \text{Pl}_v(\mathcal{Y}^c) \sum_{j \in \mathcal{ICCPF}_v} m_X(\delta_j) \] (3.25)
The summations in Eqs. (3.22)–(3.25) provide formulas by which the CBFs, CCBFs, CPFs and CCPFs defined in Eqs. (3.14)–(3.17) can be calculated, although determination of the sets in Eqs. (3.18)–(3.21) can be difficult.

As discussed in conjunction with Eqs. (2.13)–(2.18), the probability space \( (\mathcal{X}, \mathcal{X}, p_x) \) characterizing the uncertainty in a vector \( x \) is typically constructed from probability spaces \( (\mathcal{X}, \mathcal{X}, p_x) \) characterizing the uncertainty in the elements \( x_i \) of \( x \). A similar situation holds for evidence theory in which an evidence space \( (\mathcal{X}, \mathcal{X}, m_x) \) characterizing the uncertainty in a vector \( x \) is constructed from evidence spaces \( (\mathcal{X}, \mathcal{X}, m_x) \) characterizing the uncertainty in the elements \( x_i \) of \( x \). Specifically, \( \mathcal{X} \) and \( \mathcal{X} \) for the evidence space \( (\mathcal{X}, \mathcal{X}, m_x) \) are defined as in Eqs. (2.14) and (2.15), and \( m_x \) is defined by
\[ m_x(\delta) = \begin{cases} \prod_{i=1}^{n} m_x(\delta_i) & \text{if } \delta = \delta_1 \times \delta_2 \times \cdots \times \delta_n \in \mathcal{X} \\ 0 & \text{otherwise} \end{cases} \] (3.26)
under the assumption that the \( x_i \) are independent. The development is more complex when the \( x_i \) are not independent. For a vector \( x \) of the form defined in Eq. (2.13), the structure of \( \mathcal{X} \) for the evidence space \( (\mathcal{X}, \mathcal{X}, m_x) \) is much simpler than the structure of \( \mathcal{X} \) for an analogous probability space \( (\mathcal{X}, \mathcal{X}, p_x) \). In particular, the set \( \mathcal{X} \) for the evidence space \( (\mathcal{X}, \mathcal{X}, m_x) \) is the same as the set \( \mathcal{C} \) in Eq. (2.15); in contrast, the set \( \mathcal{X} \) for an analogous probability space \( (\mathcal{X}, \mathcal{X}, p_x) \) is constructed from \( \mathcal{C} \) and in general contains an uncountably infinite number of elements rather than the finite number of elements usually contained in \( \mathcal{C} \) (Ref. [52], Section IV.6; Ref. [53], Section 2.6).

As indicated in Eq. (2.12), a Monte Carlo calculation of CDFs and CCDFs is possible. Indeed, this is probably the most widely used procedure for their calculation in real problems. A Monte Carlo calculation of CBFs, CCBFs, CPFs and CCPFs is also possible [60,61]. However, the details of this calculation as outlined in Table 1 are not as succinct as the probabilistic calculation indicated in Eq. (2.12).

Step 1 of the approximation procedure in Table 1 involves the definition of a distribution for use in sampling. As long as \( f \) is continuous and the distribution is defined so that every point in \( \mathcal{X} \) has the potential to be sampled, the actual definition of the distribution is unimportant in the sense that the Monte Carlo procedure will converge towards the correct CBF, CCBF, CCPF and CCBF as the sample size

| Table 1 |
| Monte Carlo approximation of the CPF, CBF, CCPF and CCBF (see Eqs. (3.14)–(3.17)) for the range of a function \( f \) defined on the sample space \( \mathcal{X} \) of an evidence space \( (\mathcal{X}, \mathcal{X}, m_x) \) |

| Step 1. Define a probability distribution (e.g. uniform) on \( \mathcal{X} \) for use in generating a sample from \( \mathcal{X} \). |
| Step 2. Generate a random or Latin hypercube sample \( x_i, k = 1, 2, \ldots, nS \), from \( \mathcal{X} \) in consistency with the distribution defined in step 1. |
| Step 3. Evaluate \( f \) to create the mapping \( y_i = f(x_i), k = 1, 2, \ldots, nS \), from the domain (i.e. \( \mathcal{X} \)) to the range (i.e. \( \mathcal{Y} = f(\mathcal{X}) \)) of \( f \). |
| Step 4. Estimate CPF, CBF, CCPF, CCBF by |
| \( \text{CPF} = \{y_i, \text{Pl}_x([x : y_i \leq y])\}, k = 1, 2, \ldots, nS \) |
| \( \text{CBF} = \{y_i, 1 - \text{Pl}_x([x : y_i > y])\}, k = 1, 2, \ldots, nS \) |
| \( \text{CCPF} = \{y_i, 1 - \text{Pl}_x([x : y_i > y])\}, k = 1, 2, \ldots, nS \) |
| \( \text{CCBF} = \{y_i, 1 - \text{Pl}_x([x : y_i > y])\}, k = 1, 2, \ldots, nS \) |
increases without bound, although the rate of convergence may be significantly affected by the choice of sampling distribution. Technically, the preceding statement is not completely true. If a subset $\mathcal{X}$ satisfies $m_\mathcal{X}(\mathcal{X}) > 0$, then the probability space $(\mathcal{X}, \mathcal{X}, m_\mathcal{X})$ associated with the sampling distribution must satisfy $p_m(\mathcal{X}) > 0$; otherwise, the sampling will miss $\mathcal{X}$. For example, this would be the case if $\mathcal{X} = \{x : 0 \leq x \leq 1\}$, $\mathcal{X} = [0.5]$, $\mathcal{X}(\mathcal{X}) = 0.25$, and $p_m$ corresponds to a uniform distribution on $\mathcal{X}$.

When $(\mathcal{X}, \mathcal{X}, m_\mathcal{X})$ is constructed as indicated in conjunction with Eq. (3.26) from the evidence spaces $(\mathcal{X}_i, \mathcal{X}_i, m_i)$, $i = 1, 2, \ldots, n$, for each element $x_i$ of $\mathcal{X}$, one approach is to use $(\mathcal{X}_i, \mathcal{X}_i, m_i)$ to define a sampling distribution $(\mathcal{X}_i, \mathcal{X}_i, p_i)$ for each $x_i$ and then define the sampling distribution for $\mathcal{X}$ on the basis of the sampling distributions for the $x_i$ as indicated in conjunction with Eq. (2.18). Specifically, the sets $\mathcal{X}_ij$, $j = 1, 2, \ldots, m(i) \leq \infty$, contained in $\mathcal{X}_i$ for each evidence space $(\mathcal{X}_i, \mathcal{X}_i, m_i)$ can be treated as discrete outcomes with probabilities $p_i(\mathcal{X}_ij) = m_i(\mathcal{X}_ij)$ (Note: The $\mathcal{X}_i$ associated with each evidence space $(\mathcal{X}_i, \mathcal{X}_i, p_i)$ must be countable for the requirement $\sum_{\mathcal{X}_i \in \mathcal{X}} m_i(\mathcal{X}_i) = 1$ to be satisfied; thus, the indicated countability is always present). Further, conditional on its occurrence, a uniform (or some other) distribution over $\mathcal{X}_ij$ can be assumed. These two assumptions (i.e. $p_i(\mathcal{X}_ij) = m_i(\mathcal{X}_ij)$ and a specified distribution over $\mathcal{X}_ij$ given its occurrence) are sufficient to define a probability distribution (i.e. probability space $(\mathcal{X}_ij, \mathcal{X}_ij, p_ij)$) over $\mathcal{X}_i$. For example, if $d_{ij}$ is the density function defined on $\mathcal{X}_ij$ conditional on the occurrence of $\mathcal{X}_ij$, then the corresponding density function $d_i$ associated with $(\mathcal{X}_i, \mathcal{X}_i, p_i)$ is given by

$$d_i(x) = \sum_{j=1}^{m(i)} p_i(\mathcal{X}_ij)d_{ij}(x) = \sum_{j=1}^{m(i)} m_i(\mathcal{X}_ij)d_{ij}(x) \tag{3.27}$$

for $x \in \mathcal{X}_i$ and with the convention that $d_{ij}(x) = 0$ for $x \notin \mathcal{X}_ij$.

In Step 2, a sample is generated from $\mathcal{X}$ in consistency with the distribution over $\mathcal{X}$ defined in Step 1. For simple random sampling, the distribution defined by $(\mathcal{X}, \mathcal{X}, p_\mathcal{X})$ can be sampled by using two uniform random numbers from $[0, 1]$, denoted $r_1$ and $r_2$, to obtain the sampled value for $x_i$ in each sample element $x_i$. Specifically, the first number, $r_1$, is used to select a set $\mathcal{X}_ij$ with probability $p_i(\mathcal{X}_ij)$, and the second number, $r_2$, is used to select a value $x_{ik}$ from $\mathcal{X}_ij$ in consistency with the definition of the density function $d_{ij}$. This approach requires $2(m(n))$ random numbers to generate the sample in Step 2 of Table 1. The specifics of random number generation have been widely studied and are outside the scope of this presentation [62–65]. Another possibility is to actually construct the density functions indicated in Eq. (3.27) and then use Latin hypercube sampling [66–68]. The efficient stratification properties associated with Latin hypercube sampling have the potential to reduce the number of function evaluations required to obtain a given level of resolution in the estimation of CPFs, CBFs, CCPFs and CCBFs.

The function $f$ is evaluated in Step 3 for the sample elements generated in Step 2. For nontrivial functions (i.e. computer programs whose execution requires significant computational resources), this step is the most computationally demanding part of the procedure and can significantly limit the size of the sample that can be used.

Estimates for CPFs, CBFs, CCPFs and CCBFs are obtained in Step 4. Due to the subset relationship involved in the definition of belief (see Eq. (3.3)), beliefs cannot be estimated directly in a Monte Carlo simulation as is the case for plausibilities. Rather, beliefs must be estimated indirectly with use of the identity in Eq. (3.5), which is the reason for the form of the estimated CBF and CCBF given in Step 4. Once obtained, the estimated CPFs, CBFs, CCPFs and CCBFs can be displayed as plots of the form appearing in Fig. 2 with the plotting convention that a curve is represented by vertical and horizontal line segments between defined points.

Discussions of related, but not identical, Monte Carlo procedures for use in conjunction with evidence theory are available in several presentations [69–72].

Evidence theory is not really distinct from probability theory. Rather, evidence theory can be viewed as a special application of probability theory. In particular, the specification of a triple $(\mathcal{X}, \mathcal{X}, p)$ in evidence theory can be viewed as providing an incomplete specification of the triple $(\mathcal{X}, \mathcal{X}, p)$ in probability theory. As previously noted, $\mathcal{S}$ is generally not the same set in the two theories. There are many possible ways in which $(\mathcal{X}, \mathcal{X}, p)$ could be developed in a manner that is consistent with the information provided in $(\mathcal{X}, \mathcal{X}, p)$. For a subset $\mathcal{X}$ of $\mathcal{X}$, the belief $\text{Bel}(\mathcal{X})$ is the smallest possible probability that could be assigned to $\mathcal{X}$ in such a development without contradicting either the rules of probability theory or the information contained in $(\mathcal{X}, \mathcal{X}, p)$, and the plausibility $\text{Pl}(\mathcal{X})$ is the largest possible probability that could be assigned to $\mathcal{X}$ in such a development without contradicting either the rules of probability theory or the information contained in $(\mathcal{X}, \mathcal{X}, p)$. The preceding view of evidence theory is consistent with the early development work of Dempster [57–59]. Some individuals prefer to reverse the preceding perspective and view probability theory as a special case of evidence theory. This view, which is most appropriate for finite sample spaces, results from the conceptualization of a probability space as being a degenerate evidence space in which the belief and plausibility of a set are the same and equal to the probability of that set. In contrast, Shafer, who extended Dempster’s work, views evidence theory as a logic independent of probability theory for reasoning under uncertainty [19]. The key axiom in this logic is the equality in Eq. (3.5), which requires that the plausibility of something being true must equal one minus the belief in it not being true.
Additional information on evidence theory is available in Refs. [18–24] and in the references cited by these publications.

4. Possibility theory

Possibility theory [25–29] provides another alternative to probability theory for the representation of uncertainty. Like evidence theory, possibility theory involves two specifications of likelihood, a necessity and a possibility, for each subset of the universal set under consideration. While evidence theory is closely tied to probability theory, possibility theory is more closely tied to fuzzy set theory.

Formally, an application of possibility theory involves the specification of a pair $(\mathcal{S}, r)$, where (i) $\mathcal{S}$ is a set that contains everything that could occur in the particular universe under consideration, and (ii) $r$ is a function defined on $\mathcal{S}$ such that $0 \leq r(x) \leq 1$ for $x \in \mathcal{S}$ and $\sup\{r(x) : x \in \mathcal{S}\} = 1$. The function $r$ provides a measure of the likelihood that can be assigned to each element of the universal set (i.e. sample space) $\mathcal{S}$ and is referred to as a possibility distribution function. In consistency with the designation of $(\mathcal{S}, \mathcal{S}, \mathcal{P})$ and $(\mathcal{S}, \mathcal{S}, \mathcal{M})$ as probability and evidence spaces, respectively, the pair $(\mathcal{S}, r)$ is referred to as a possibility space.

The possibility, $\text{Pos}(\mathcal{S})$, and necessity, $\text{Nec}(\mathcal{S})$, for a subset $\mathcal{E} \subseteq \mathcal{S}$ are defined by

$$\text{Pos}(\mathcal{E}) = \sup\{r(x) : x \in \mathcal{E}\}$$

and

$$\text{Nec}(\mathcal{E}) = 1 - \sup\{r(x) : x \in \mathcal{E}^c\} = 1 - \text{Pos}(\mathcal{E}^c).$$

In words but without formally acknowledging the technical need to consider least upper bounds (i.e. sup’s) when infinite sets are under consideration, $\text{Pos}(\mathcal{S})$ is the largest value of $r(x)$ for $x \in \mathcal{E}$. Thus, the definitions of $\text{Pos}(\mathcal{S})$ and $\text{Nec}(\mathcal{S})$ derive from properties of individual elements of a universal set; in contrast, probability, plausibility and belief are defined in terms of subsets of a universal set.

Relationships satisfied by possibility and necessity include

\begin{align*}
1 &= \text{Nec}(\mathcal{S}) + \text{Pos}(\mathcal{S}) \quad \text{(4.3)} \\
nec(\mathcal{S}) &\leq \text{Pos}(\mathcal{S}) \quad \text{(4.4)} \\
1 &\leq \text{Pos}(\mathcal{E}) + \text{Pos}(\mathcal{E}^c) \quad \text{(4.5)} \\
1 &\geq \text{Nec}(\mathcal{S}) + \text{Nec}(\mathcal{S}^c) \quad \text{(4.6)} \\
1 &= \max\{\text{Pos}(\mathcal{S}), \text{Pos}(\mathcal{S}^c)\} \quad \text{(4.7)} \\
0 &= \min\{\text{Nec}(\mathcal{S}), \text{Nec}(\mathcal{S}^c)\} \quad \text{(4.8)} \\
\text{Pos}(\mathcal{E}) < 1 &\Rightarrow \text{Nec}(\mathcal{E}) = 0 \quad \text{(4.9)} \\
\text{Nec}(\mathcal{S}) > 0 &\Rightarrow \text{Pos}(\mathcal{S}) = 1 \quad \text{(4.10)}
\end{align*}

(see Ref. [73], p. 34). The relationships in Eqs. (4.3)–(4.6) are analogous to relationships that also hold for plausibility and belief. However, the relationships in Eqs. (4.7)–(4.10) do not, in general, hold for plausibility and belief.

In general, evidence theory and possibility theory provide conceptually different representations of uncertainty. However, in the special case where the elements of the set $\mathcal{S}$ in an evidence space $(\mathcal{S}, \mathcal{S}, \mathcal{M})$ are nested, the definition of $r$ by

$$r(x) = \mathcal{P}(\{x\})$$

for $x \in \mathcal{S}$ results in the possibilities and necessities of subsets of $\mathcal{S}$ defined with $r$ being the same as the plausibilities and beliefs of subsets of $\mathcal{S}$ defined with $m$.

As indicated in conjunction with Eqs. (3.8)–(3.11), CBFs, CCBFs, CPFs and CCPFs can be used to provide summaries of the information contained in an evidence space $(\mathcal{S}, \mathcal{S}, \mathcal{M})$. Similarly, cumulative necessity functions (CNFs), complementary cumulative necessity functions (CCNFs), cumulative possibility functions (CPoFs), and complementary cumulative possibility functions (CCCPoFs) can be used to summarize the necessities and possibilities associated with a possibility space $(\mathcal{S}, r)$ when $\mathcal{S}$ contains real numbers. Specifically, CNFs, CCNFs, CPoFs and CCPoFs are defined by the sets of points

\begin{align*}
\mathcal{CNF} &= \{(v, \text{Nec}(\mathcal{S}_v)), v \in \mathcal{S}\} \quad \text{(4.12)} \\
\mathcal{CCNF} &= \{(v, \text{Nec}(\mathcal{S}_v)), v \in \mathcal{S}\} \quad \text{(4.13)} \\
\mathcal{CPF} &= \{(v, \text{Pos}(\mathcal{S}_v)), v \in \mathcal{S}\} \quad \text{(4.14)} \\
\mathcal{CCCPF} &= \{(v, \text{Pos}(\mathcal{S}_v)), v \in \mathcal{S}\} \quad \text{(4.15)}
\end{align*}

where $\mathcal{S}_v$ is defined in Eq. (2.4). Plots of the points in the preceding sets produce CNFs, CCNFs, CPoFs and CCPoFs, respectively (Fig. 3). All the properties of necessity and possibility listed in Eqs. (4.3)–(4.10) can be observed by examining the results in Fig. 3.

As previously discussed, probability can be used to represent the uncertainty in function evaluations when the uncertainty associated with the domain of the function is characterized by a probability space $(\mathcal{S}, \mathcal{S}, \mathcal{P})$ (see Eq. (2.5)), and belief and plausibility can be used to represent the uncertainty in function evaluations when the uncertainty associated with the domain of the function is characterized by an evidence space $(\mathcal{S}, \mathcal{S}, \mathcal{M})$ (see Eqs. (3.12) and (3.13)). Similarly, necessity and possibility can be used to represent the uncertainty in function evaluations when the uncertainty associated with the domain of the function is characterized by a possibility space $(\mathcal{S}, r)$.

For the function $f$ in Eq. (2.5) and a possibility space $(\mathcal{S}, r)$, the range of $f$ is given by the set $\mathcal{Y}$ defined in Eq. (2.6). Analogously to the cases for probability and evidence theory, the uncertainty in the values of $y$ contained in $\mathcal{Y}$ derives from the possibility space $(\mathcal{S}, r_x)$ that characterizes the uncertainty in $x$ and from the properties of the function $f$.

In concept, $(\mathcal{S}, r)$ and $f$ induce a possibility space $(\mathcal{Y}, r_y)$. In practice, $r_y$ is not determined. Rather, the possibility $\text{Pos}_y(\mathcal{E})$ and necessity $\text{Nec}_y(\mathcal{E})$ for a subset $\mathcal{E}$ of $\mathcal{Y}$ are determined from the distribution function $r_y$ associated with
\((\mathcal{X}, r_X)\). Specifically,

\[
\text{Pos}_Y(\delta) = \text{Pos}_X(f^{-1}(\delta)) = \sup \{r_X(x) : x \in f^{-1}(\delta)\}
\]

and

\[
\text{Nec}_Y(\delta) = \text{Nec}_X(f^{-1}(\delta)) = 1 - \text{Pos}_X([f^{-1}(\delta)]^c),
\]

where \(f^{-1}(\delta)\) is defined in Eq. (2.8). The preceding definitions for \(\text{Nec}_Y(\delta)\) and \(\text{Pos}_Y(\delta)\) are analogous to the definition of \(p_Y(\delta)\) in Eq. (2.7) and the definitions of \(\text{Bel}_Y(\delta)\) and \(\text{Pl}_Y(\delta)\) in Eqs. (3.12) and (3.13).

As discussed in conjunction with Eqs. (2.9)–(2.11) for probability theory and Eqs. (3.14)–(3.17) for evidence theory, the uncertainty in \(y\) characterized by the possibility space \((\mathcal{Y}, r_Y)\) can be summarized with CNFs, CCNFs, CPoFs and CCPoFs. In particular, the CNFs, CCNFs, CPoFs and CCPoFs for \(y\) are defined by the sets of points

\[
\mathcal{C}_Y = \{[v, \text{Nec}_Y(\delta)_v^2], v \in \mathcal{Y}\}
\]

\[
\mathcal{C}_Y = \{[v, \text{Nec}_Y(\delta)_v], v \in \mathcal{Y}\}
\]

\[
\mathcal{E}_Y = \{[v, \text{Pos}_Y(\delta)_v], v \in \mathcal{Y}\}
\]

\[
\mathcal{E}_Y = \{[v, \text{Pos}_Y(\delta)_v], v \in \mathcal{Y}\}
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\mathcal{E}_Y = \{[v, \text{Pos}_Y(\delta)_v], v \in \mathcal{Y}\}
\]

where \(\mathcal{Y}_v\) is defined in Eq. (2.11). Plots of the points contained in \(\mathcal{C}_Y\), \(\mathcal{C}_Y\), \(\mathcal{E}_Y\) and \(\mathcal{E}_Y\) produce a figure similar to Fig. 3 and provide a visual representation of the uncertainty in \(y\) in terms of necessity and plausibility.

The possibility space \(\mathcal{X}, r_X\) for a vector \(\mathbf{x} = [x_1, x_2, \ldots, x_n]\) is often constructed from the possibility spaces \((\mathcal{X}_i, r_{X_i})\) for the elements \(x_i\) of \(\mathbf{x}\) in a manner analogous to that previously described for probability spaces (see Eqs. (2.13)–(2.17)) and evidence spaces (see Eq. (3.26)). Specifically, \(\mathcal{X}\) is defined as in Eq. (2.14), and \(r_{X}\) is defined by

\[
r_X(x) = \min\{r_{X_1}(x_1), r_{X_2}(x_2), \ldots, r_{X_n}(x_n)\}
\]

for \(x \in \mathcal{X}\) and under the assumption that the \(x_i\) are independent. The development is more complex when the \(x_i\) are not independent.

As previously discussed for CDFs and CCDFs in probability theory (Eq. (2.12)) and CBFs, CCBFs, CPFs and CCPFs in evidence theory (Table 1), Monte Carlo procedures can be used to estimate CNFs, CCNFs, CPoFs and CCPoFs (Table 2). The procedure for CNFs, CCNFs, CPoFs and CCPoFs is the same as the procedure for CBFs, CCBFs, CPFs and CCPFs except for the estimation of necessities and possibilities rather than beliefs and plausibilities at the final step (i.e. Step 4).

Additional information on possibility theory is available in Refs. [25–29] and in the references cited by these publications.

### Table 2

Monte Carlo approximation of the CPoF, CNF, CCPoF and CCNF (see Eqs. (4.18)–(4.21)) for the range of a function \(f\) defined on the sample space \(\mathcal{X}\) of a possibility space \((\mathcal{X}, r_X)\)

**Steps 1, 2, 3.** Same as steps 1, 2, 3 in Table 1

**Step 4.** Estimate CPoFs, CNFs, CCPoFs, CCNFs by \(\mathcal{E}_Y\)

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5. Interval analysis

Interval analysis [30–35] is based on using algebraic procedures, typically implemented through appropriate software, to propagate intervals of possible values for variables through to an interval of possible values for a function of these variables. Thus, in the notation previously used, the uncertainty in each element $x_i$ of $x$ would be represented by an interval, and the goal of interval analysis would be to construct the smallest interval that exactly contained the resultant possible values for $f(x)$.

Unlike applications based on probability, evidence and possibility theory, interval analysis does not attempt to infer an uncertainty structure on $f(x)$ based on an uncertainty structure assumed for $x$. Thus, the uncertainty representation for $f(x)$ obtained with interval analysis lacks the structure obtained with the other indicated uncertainty representations. For this reason, interval analysis provides uncertainty representations that are different in spirit from those obtained with probability, evidence and possibility theory.

Because of the preceding difference, interval analysis will not be extensively considered in this presentation. However, the range for $f(x)$ obtained with the Monte Carlo techniques described for use in conjunction with probability, evidence and possibility theory provides an estimate of the interval analysis solution for the uncertainty in $f(x)$. Further, this approximate solution appropriately incorporates the effects of repeated appearances of the $x_i$ in the evaluation of $f(x)$ into the estimated range of $f(x)$.

6. Algebraic problem set: epistemic uncertainty

The algebraic problem set proposed by Oberkampf et al. [51] involves the simple model
\[ y = f(a, b) = (a + b)^r, \]  \hspace{1cm} (6.1)
where $a$ and $b$ are uncertain parameters. Problems 1, 2 and 3 are assumed to involve only epistemic uncertainty in $a$ and $b$. Specifically, ranges of possible values of $a$ and $b$ are given by one or more independent sources of information (Table 3).

In all problems, the possible values for $a$ and $b$ are contained within the intervals $[0.1, 1.0]$ and $[0.0, 1.0]$, respectively. The corresponding values of $y$ can be displayed as a surface plot (Fig. 4a) or as a contour plot (Fig. 4b). The function $f$ is continuous and single-valued on $[0.1, 1.0] \times [0.0, 1.0]$, but the inverse of $f$ is not single valued as indicated by the contour lines in Fig. 4.

The uncertainty in $y$ that derives from the uncertainty in $a$ and $b$ described in Table 3 will be represented with probability theory, evidence theory and possibility theory. These representations will require the introduction of probability spaces, evidence spaces and possibility spaces for $a$ and $b$, the corresponding spaces that result for the vector $x = [a, b]$, and finally, approximations to the resultant spaces for $y$.

In each problem summarized in Table 3, $nA$ and $nB$ independent sources provide uncertainty ranges for $a$ and $b$ (i.e. $nA = 1$ for Problems 1, 2a, 2b, 2c; $nA = 3$ for Problems 3a, 3b, 3c; $nB = 1$ for Problem 1; $nB = 4$ for Problems 2a, 2b, 2c, 3a, 3b, 3c). Specifically, intervals
\[ A_r = [a_{\min,r}, a_{\max,r}], \quad r = 1, 2, ..., nA, \]  \hspace{1cm} (6.2)
\[ B_r = [b_{\min,r}, b_{\max,r}], \quad s = 1, 2, ..., nB, \]  \hspace{1cm} (6.3)
of possible values are specified for $a$ and $b$. For consistency with the notation used in Sections 3–5, the preceding intervals will be represented by the sets
\[ \mathcal{A}_r = \{ a : a_{\min,r} \leq a \leq a_{\max,r} \}, \quad r = 1, 2, ..., nA, \]  \hspace{1cm} (6.4)
and
\[ \mathcal{B}_s = \{ b : b_{\min,s} \leq b \leq b_{\max,s} \}, \quad s = 1, 2, ..., nB. \]  \hspace{1cm} (6.5)

No information is provided on the likelihoods of $a$ and $b$ within the indicated intervals.

The initial problem that must be addressed in developing uncertainty representations for $y$ is how to define uncertainty representations for $a$ and $b$ on the basis of the available information. There exists an extensive literature on the aggregation of multiple sources of information, and a variety of aggregation procedures have been proposed [74–77]. For this presentation, the view is taken that the total range of uncertainty should be preserved and that, in some sense, the amount of likelihood assigned to a specific
value of \(a\) or \(b\) should reflect the number of sources of information (e.g. experts) that specified an interval containing that value.

Consistent with the desire to preserve the total range of uncertainty for \(a\) and \(b\), the sample spaces (i.e. universal sets) for \(a\) and \(b\) in each problem are defined by

\[
\mathcal{A} = \bigcup_{i=1}^{nA} \mathcal{A}_i \quad \text{and} \quad \mathcal{B} = \bigcup_{j=1}^{nB} \mathcal{B}_j. \tag{6.6}
\]

The sample spaces \(\mathcal{A}\) and \(\mathcal{B}\) remain unchanged in subsequent definitions of probability, evidence and possibility spaces for \(a\) and \(b\).

Probability spaces \((\mathcal{A}, \bar{\mathcal{A}}, p_A)\) and \((\mathcal{B}, \bar{\mathcal{B}}, p_B)\) for \(a\) and \(b\) are defined by assigning uniform distributions over each of the sets \(\mathcal{A}_i\) and \(\mathcal{B}_j\). These distributions are conditional on the occurrence (i.e. specification by a particular source) of each interval. As is almost always the case in the definition of probability spaces, no attempt is made to formally define \(\bar{\mathcal{A}}\) and \(\bar{\mathcal{B}}\), and \(p_A\) and \(p_B\) are defined indirectly by the specification of density functions \(d_A\) and \(d_B\) defined on \(\mathcal{A}\) and \(\mathcal{B}\). Specifically,

\[
d_A(a) = \sum_{i=1}^{nA} d_{A_i}(a)/nA \tag{6.7}
\]

for \(a \in \mathcal{A}\) and

\[
d_B(b) = \sum_{j=1}^{nB} d_{B_j}(b)/nB \tag{6.8}
\]

for \(b \in \mathcal{B}\), where, except for \(\mathcal{B}_1\) in Problems 3a and 3b, the density functions

\[
d_{A_i}(a) = \begin{cases} 1/(a_{\max,r} - a_{\min,r}) & \text{if } a \in \mathcal{A}_i \\ 0 & \text{otherwise} \end{cases} \tag{6.9}
\]

and

\[
d_{B_j}(b) = \begin{cases} 1/(b_{\max,s} - b_{\min,s}) & \text{if } b \in \mathcal{B}_j \\ 0 & \text{otherwise} \end{cases} \tag{6.10}
\]

define distributions that are uniform on \(\mathcal{A}_i\) and \(\mathcal{B}_j\) and assign probabilities of zero elsewhere. Problems 3a and 3b are a special case in that \(\mathcal{B}_1\) contains a single value (i.e. \(b = 0.6\)), and as a result,

\[
d_{B_1}(b) = \delta(b - 0.6), \tag{6.11}
\]

where \(\delta\) is the Dirac delta function (i.e. the function for which \(\delta(v) = 0\) if \(v \neq 0\) and whose integral from \(-\infty\) to \(\infty\) is 1).

Given the preceding definitions of \(d_A\) and \(d_B\),

\[
p_A(\mathcal{U}) = \int_{\mathcal{U}} d_A(a)da \tag{6.12}
\]

and

\[
p_B(\mathcal{V}) = \int_{\mathcal{V}} d_B(b)db \tag{6.13}
\]

for any reasonable subsets \(\mathcal{U}\) and \(\mathcal{V}\) of \(\mathcal{A}\) and \(\mathcal{B}\), respectively (i.e. sets that satisfy the conditions to be in \(\bar{\mathcal{A}}\) and \(\bar{\mathcal{B}}\)). The resultant probability spaces \((\mathcal{A}, \bar{\mathcal{A}}, p_A)\) and \((\mathcal{B}, \bar{\mathcal{B}}, p_B)\) will be used to obtain a probabilistic representation of the uncertainty in \(y\) and also in Monte Carlo simulations to determine representations of the uncertainty in \(y\) with evidence theory and possibility theory.

Evidence spaces \((\mathcal{A}, \bar{\mathcal{A}}, m_A)\) and \((\mathcal{B}, \bar{\mathcal{B}}, m_B)\) for \(a\) and \(b\) are defined by assuming that the sets \(\mathcal{A}_r\), \(r = 1, 2, \ldots, nA\), and \(\mathcal{B}_s\), \(s = 1, 2, \ldots, nB\), are the focal elements of these spaces. Specifically, \(\mathcal{A}\) and \(\mathcal{B}\) are defined as indicated in Eq. (6.6), and \(\bar{\mathcal{A}}\) and \(\bar{\mathcal{B}}\) are defined by

\[
\bar{\mathcal{A}} = \{\mathcal{A}_r : r = 1, 2, \ldots, nA\} \tag{6.14}
\]
where \( B = \{ \mathcal{B}_s : s = 1, 2, \ldots, n_B \} \).

Further, in consistency with the guidance in Ref. [51] that all sources of information are equally credible, the BPA assigned to a particular focal element is assumed to be the fraction of the sources that specified that focal element. Specifically,

\[
m_A(\mathcal{A}_r) = N(\mathcal{A}_r)/n_A \tag{6.16}
\]

and

\[
m_B(\mathcal{B}_s) = N(\mathcal{B}_s)/n_B \tag{6.17}
\]

where \( N(\mathcal{A}_r) \) and \( N(\mathcal{B}_s) \) are the number of sources that specified the sets \( \mathcal{A}_r \) and \( \mathcal{B}_s \), respectively. In the example problems, \( N(\mathcal{A}_r) \) and \( N(\mathcal{B}_s) \) are always 1.

Possibility spaces \((\mathcal{A}_r, r_A)\) and \((\mathcal{B}_s, r_B)\) for \( a \) and \( b \) are defined by initially assuming that the values of the distribution functions \( r_A \) and \( r_B \) should correspond to the fraction of the sources that specified given values for \( a \) and \( b \). This convention is consistent with the specification that all sources of information are equally credible and produces the definitions

\[
r_A(a) = \sum_{i=1}^{n_A} \delta_{A_i}(a)/n_A \tag{6.18}
\]

for \( a \in \mathcal{A}_r \), and

\[
r_B(b) = \sum_{j=1}^{n_B} \delta_{B_j}(b)/n_B \tag{6.19}
\]

for \( b \in \mathcal{B}_s \), where

\[
\delta_{A_i}(a) = \begin{cases} 1 & \text{if } a \in \mathcal{A}_r, \\ 0 & \text{otherwise}, \end{cases} \tag{6.20}
\]

\[
\delta_{B_j}(b) = \begin{cases} 1 & \text{if } b \in \mathcal{B}_s, \\ 0 & \text{otherwise}, \end{cases} \tag{6.21}
\]

and \( \mathcal{A}_r \) and \( \mathcal{B}_s \) are defined in Eq. (6.6).

The preceding definitions for \( r_A \) and \( r_B \) result in possibility spaces when the sets \( \mathcal{A}_r \), \( r = 1, 2, \ldots, n_A \), and \( \mathcal{B}_s \), \( s = 1, 2, \ldots, n_B \), are consistent; that is, when

\[
\bigcap_{r=1}^{n_A} \mathcal{A}_r \neq \phi \quad \text{and} \quad \bigcap_{s=1}^{n_B} \mathcal{B}_s \neq \phi. \tag{6.22}
\]

However, when the sets are not consistent, there is no value in the sample space (i.e. \( \mathcal{A}_r \) or \( \mathcal{B}_s \)) at which the distribution function (i.e. \( r_A \) or \( r_B \)) equals 1. Thus, the definition of a possibility space is not satisfied (see Section 4). As an aside, because \( r_A \) and \( r_B \) assume only a finite number of values in the problems under consideration, there is no need to consider sup’s as indicated in Eqs. (4.1) and (4.2) in the definition of a possibility space. Such inconsistency exists for \( a \) in Problem 3c and for \( b \) in Problems 2c and 3c (see Table 3). Thus, the approach given in the preceding paragraph defines possibility spaces \((\mathcal{A}_r, r_A)\) for \( a \) for all problems except Problem 3c and possibility spaces \((\mathcal{B}_s, r_B)\) for \( b \) for all problems except Problems 2c and 3c.

Given that \( r_A \) and \( r_B \) in Eqs. (6.18) and (6.19) were developed in a manner consistent with the guidance that all sources of information are equally credible, it is desirable to preserve the information that they contain in developing possibility spaces for \( a \) and \( b \). The approach employed in this presentation is to scale \( r_A \) and \( r_B \) to produce new distribution functions that satisfy the requirement of equality to 1 for one or more elements of the sample space. The scaling can be either additive or multiplicative. For notational convenience, let \( r \) denote \( r_A \) or \( r_B \) for a problem in which the corresponding \( \mathcal{A}_r, s \) or \( \mathcal{B}_s \) are not consistent. New distribution functions \( r_{\text{add}} \) and \( r_{\text{mult}} \) based on additive and multiplicative scaling, respectively, are given by

\[
r_{\text{add}}(v) = r(v) + \min \{ 1 - r(u) : u \in \mathcal{A}_r \} \tag{6.23}
\]

and

\[
r_{\text{mult}}(v) = r(v)/\max \{ r(u) : u \in \mathcal{A}_r \}, \tag{6.24}
\]

where \( v \in \mathcal{A}_r \) and \( \mathcal{A}_r \) corresponds to \( \mathcal{A}_r \) or \( \mathcal{B}_s \) depending on whether \( r_A \) or \( r_B \) is being scaled. (Note: In general, inf and sup rather than min and max would be used in Eqs. (6.23) and (6.24) but this generality is not needed for the problems under consideration). For \( a \) in Problem 3a, the indicated scalings result in two possible values for the possibility space \((\mathcal{A}_r, r_A)\) and \((\mathcal{B}_s, r_B)\), where \( r_{\text{add}} \) and \( r_{\text{mult}} \) are defined as indicated in Eqs. (6.23) and (6.24), respectively. Similarly for \( b \) in Problems 2c and 3c, the scalings result in two possible values for \((\mathcal{B}_s, r_B)\) and \((\mathcal{B}_s, r_{\text{mult}})\) and \((\mathcal{B}_s, r_{\text{mult}})\).

Now that probability, evidence and possibility spaces have been constructed for \( a \) and \( b \), the corresponding spaces can be constructed for \( x = [a, b] \). For probability theory, the probability spaces \((\mathcal{A}, \mathcal{B}, p_A)\) and \((\mathcal{B}, \mathcal{B}, p_B)\) and their corresponding density functions \( d_A \) and \( d_B \) are known and give rise to a probability space \((\mathcal{A}, \mathcal{B}, p_X)\) for \( x \). Specifically, the sample space \( \mathcal{X} \) is given by

\[
\mathcal{X} = \mathcal{A} \times \mathcal{B} = \{ x : x = [a, b], a \in \mathcal{A}, b \in \mathcal{B} \} \tag{6.25}
\]

(see Eq. (2.14)), and the density function \( d_X \) is defined by

\[
d_X(x) = d_A(a)d_B(b) \tag{6.26}
\]

for \( x = [a, b] \in \mathcal{X} \) (see Eq. (2.18)). The set \( \mathcal{C} \) can be developed from the set

\[
\mathcal{C} = \{ \mathcal{C} : \mathcal{C} = \mathcal{U} \times \mathcal{U}, \mathcal{U} \in \mathcal{A}, \mathcal{U} \in \mathcal{B} \} \tag{6.27}
\]

(see Eq. (2.15)), and the function \( p_X \) can be formally defined by

\[
p_X(\mathcal{C}) = \int_{\mathcal{C}} d_X(x) dV = \int_{\mathcal{C}} d_A(a)d_B(b) da\, db \tag{6.28}
\]
for $\mathcal{E} \in \mathcal{X}$ (see Eq. (2.17)) and by

$$p_X(\mathcal{E}) = p_X(\mathcal{Y}|p_Y(\mathcal{Y}))$$  \hspace{1cm} (6.29)

for the special case of $\mathcal{E} = \mathcal{U} \times \mathcal{Y} \in \mathcal{X}$ (see Eq. (2.16)). A formal development of $\mathcal{X}$ and $p_X$ is typically not carried out and is not needed in the computational evaluation of the problems considered in this presentation. In practice, $p_X$ is evaluated by an integral involving $d_X$ as indicated in Eq. (6.28) or by a numerical approximation to this integral.

For evidence theory, the evidence spaces $(\mathcal{A}, \mathcal{B}, m_B)$ and $(\mathcal{A}, \mathcal{P}, m_B)$ are known. The resultant evidence space $(\mathcal{A}, m_X)$ for $b = \{a, b\}$ has $\mathcal{X}$ defined as in Eq. (6.25) and $\mathcal{X}$ and $m_X$ defined by

$$\mathcal{X} = \{\mathcal{E} : \mathcal{E} \in \mathcal{A}, \mathcal{E} \in \mathcal{A}, \mathcal{A} \in \mathcal{A}, \mathcal{B} \in \mathcal{B}\}$$  \hspace{1cm} (6.30)

(see discussion in conjunction with Eq. (3.26)) and

$$m_X(\mathcal{E}) = \left\{ \begin{array}{ll} m_B(\mathcal{A}) & for \mathcal{E} \in \mathcal{A}, \mathcal{B} \in \mathcal{X} \in \mathcal{X} \\ 0 & otherwise \end{array} \right.$$  \hspace{1cm} (6.31)

(see Eq. (3.26)). Similarly for possibility theory, the possibility spaces $(\mathcal{A}, r_A)$ and $(\mathcal{A}, r_B)$ are known. The resultant evidence space $(\mathcal{A}, r_X)$ for $x$ has $\mathcal{X}$ defined as in Eq. (6.25) and $r_X$ defined by

$$r_X(x) = \text{min}\{r_A(a), r_B(b)\}$$  \hspace{1cm} (6.32)

for $x = \{a, b\} \in \mathcal{X}$ (see Eq. (4.22)).

Now that probability, evidence and possibility spaces $(\mathcal{A}, m_X, p_X)$, $(\mathcal{A}, m_Y, p_Y)$ and $(\mathcal{A}, r_Y)$ have been introduced for the seven problems (i.e. 1, 2a, 2b, 2c, 3a, 3b, 3c) indicated in Table 3, the numerical evaluation of the corresponding uncertainty representation for the range of the function $y = f(a, b)$ defined in Eq. (6.1) is considered. In particular, the indicated spaces give rise to corresponding probability, evidence and possibility spaces $(\mathcal{A}, \mathcal{Y}, p_Y)$, $(\mathcal{A}, \mathcal{Y}, m_Y)$ and $(\mathcal{A}, \mathcal{Y}, r_Y)$ for $y$ in each of the problems. No attempt will be made to provide a complete characterization of the spaces $(\mathcal{A}, \mathcal{Y}, p_Y)$, $(\mathcal{A}, \mathcal{Y}, m_Y)$ and $(\mathcal{A}, \mathcal{Y}, r_Y)$ for each problem. Rather, the uncertainty information contained in these spaces will be summarized with CCDFs for the probability spaces $(\mathcal{A}, \mathcal{Y}, p_Y)$, CCDFs and CCPFs for the evidence spaces $(\mathcal{A}, \mathcal{Y}, m_Y)$, and CCNFs and CCPoFs for the possibility spaces $(\mathcal{A}, \mathcal{Y}, r_Y)$.

The determination of CCDFs, CCBFs and CCPFs is considered further. Further, Problem 3b is used for illustration and motivation. As previously discussed, the solution of this problem with probability theory involves probability spaces $(\mathcal{A}, \mathcal{B}, m_B)$, $(\mathcal{A}, \mathcal{B}, p_B)$, $(\mathcal{A}, \mathcal{X}, p_X)$ and $(\mathcal{Y}, \mathcal{Y}, p_Y)$ as indicated in Table 4. Similarly, the solution of this problem with evidence theory involves evidence spaces $(\mathcal{A}, \mathcal{B}, m_B)$, $(\mathcal{A}, \mathcal{X}, m_X)$ and $(\mathcal{Y}, \mathcal{Y}, m_Y)$ as indicated in Table 5. The set $\mathcal{X}$ associated with $(\mathcal{A}, \mathcal{X}, m_X)$ contains 12 sets, $\mathcal{X}_{4i-11+j} = \mathcal{A} \times \mathcal{B}$, that have nonzero BPA (Fig. 5). The CCDF associated with $(\mathcal{Y}, \mathcal{Y}, p_Y)$ (see Eq. (2.10)) can be formally defined by an appropriate integration

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<th>$\mathcal{A}_1$</th>
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Table 4
Summary of information related to probability spaces $(\mathcal{A}, \mathcal{B}, p_B)$, $(\mathcal{A}, \mathcal{X}, p_X)$ and $(\mathcal{Y}, \mathcal{Y}, p_Y)$ in Problem 3b

$$p_Y(\mathcal{Y}) = \int_{\mathcal{Y}} d_Y(y)dy = \int_{\mathcal{Y}} d_X(x)dx,$$  \hspace{1cm} (6.33)

where, as in Eq. (2.11),

$$\mathcal{Y} = \{y : y \in \mathcal{Y} and y > v\}$$  \hspace{1cm} and  \hspace{1cm} \(\mathcal{X} = \{x : x \in \mathcal{X} \}$

For a given $v$, the set $\mathcal{X}$ corresponds to the region of Fig. 4b to the right of the contour line (i.e. level curve) associated with $\mathcal{V}$. In practice, the integrals in Eq. (6.33) are difficult to define and evaluate and would also have to be defined and evaluated for a suitable number of $v$ to adequately resolve the associated CCDF. As a result and as will be done for all CCDFs in this presentation, a Monte Carlo procedure (see Eq. (2.12)) is used to estimate $p_Y(\mathcal{Y})$ and the associated CCDF. The result of applying this procedure with a sample size of $n = 10,000$ appears in Fig. 6.

In this simple example, it is possible to state and, with sufficient effort, evaluate the integrals in Eq. (6.33). Specifically, when the properties of $(\mathcal{A}, \mathcal{X}, m_X)$ and $(\mathcal{Y}, \mathcal{Y}, p_Y)$ are taken into account, the second integral in

Table 5
Summary of information related to evidence spaces $(\mathcal{A}, \mathcal{B}, m_B)$, $(\mathcal{A}, \mathcal{X}, m_X)$ and $(\mathcal{Y}, \mathcal{Y}, m_Y)$ in Problem 3b

$$m_B(\mathcal{A}) = m_B(\mathcal{A}) = m_B(\mathcal{A}) = 1/3$$  \hspace{1cm} $B = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$, $m_B(\mathcal{A}_5) = m_B(\mathcal{A}_6) = m_B(\mathcal{A}_7) = m_B(\mathcal{A}_8) = 1/4$  \hspace{1cm} $X = \mathcal{A} \times \mathcal{B}$  \hspace{1cm} $X_{4i-11+j} = \mathcal{A} \times \mathcal{B}, i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$  \hspace{1cm} $m_B(\mathcal{A}) = m_B(\mathcal{A}_i) = m_B(\mathcal{A}_j) = m_B(\mathcal{A}_k) = 1/3$  \hspace{1cm} $m_B(\mathcal{A}_i) = m_B(\mathcal{A}_j) = m_B(\mathcal{A}_k) = 1/3$  \hspace{1cm} $m_B(\mathcal{A}_i) = m_B(\mathcal{A}_j) = m_B(\mathcal{A}_k) = 1/3$  \hspace{1cm} $m_B(\mathcal{A}_i) = m_B(\mathcal{A}_j) = m_B(\mathcal{A}_k) = 1/3$
Eq. (6.33) becomes
\[ p_Y(Y_v) = \frac{1}{\min(A_v)} \left( \frac{1}{\min(B_v(a))} \right) d_y db \, da, \]  
(6.34)

where
\[ A_v = \{ a : x = [a, b] \in X_v \}, \]
\[ B_v(a) = \{ b : x = [a, b] \in X_v \}, \]
and \( X_v \) is defined in conjunction with Eq. (6.33). The integral in Eq. (6.34) is over the region to the right of the contour line for \( y = v \) in Fig. 4b and, when appropriate, the left boundary of the figure (i.e. \( \min(B_v(a)) = \max(0, v^{1/a} - a) \)). Further, the first integral in Eq. (6.33) becomes
\[ p_Y(Y_v) = \int_{\min(A_v)}^{2.0} d_f(v) dy, \]  
(6.35)

where, for an arbitrary value of \( v \) in \( \mathcal{Y} \) (i.e. in the interval [0.69, 2.0]), the density function \( f_Y(v) \) is given by
\[
\begin{align*}
\frac{d_f}{dv} & = \left[ 1 - p_Y(Y_v) \right] \\
& = \frac{d}{dv} \left[ 1 - \int_{\min(A_v)}^{1} \int_{\min(B_v(a))}^{1} d_y db \, da \right] \\
& = \frac{d}{dv} \left[ 1 - \int_{\min(A_v)}^{1} \int_{\min(B_v(a))}^{1} d_y db \, da \right] 
\end{align*}
\]
(see Ref. [78], Section 7.9). Even in this simple example, the actual closed form representation for \( d_Y(v) \) is difficult to write down due to complicating factors that result because (i) the mapping \( y = f(a, b) \) from \( X \) to \( Y \) is not one to one (see Fig. 4) and (ii) the definition of the density function \( d_Y(b) \) involves a Dirac delta function (see Table 4).

The CCBF and CCPF for \( (\mathcal{Y}, v, m_Y) \) in Problem 3b appear in Fig. 6 and were determined with the Monte Carlo procedure described in Table 1 and with the same sample of size \( nS = 10,000 \) from the probability space \( (X, \mathcal{X}, p_X) \) indicated in Table 4 used in the determination of the CCDF appearing in the figure. The probability space \( (X, \mathcal{X}, p_X) \) is consistent with the constraints implied by the evidence space \( (X, \mathcal{X}, m_X) \). As a result, the CCDF for \( y \) that derives from \( (X, \mathcal{X}, p_X) \) falls between the CCBF and CCPF for \( y \) that derive from \( (X, \mathcal{X}, m_X) \). In particular, the constraint

\[
\text{Bel}_Y(\mathcal{Y}_s) \leq P_Y(\mathcal{Y}_s) \leq \text{Pl}_Y(\mathcal{Y}_s)
\]

(6.36)

holds, with (i) \( \text{Bel}_Y(\mathcal{Y}_s) \) constituting a lower bound (technically, an infimum or ‘inf’) on the probabilities that could be assigned to \( \mathcal{Y}_s \) in a manner consistent with the definition of \( (X, \mathcal{X}, m_X) \), (ii) \( P_Y(\mathcal{Y}_s) \) providing one example of the many probabilities that could be assigned to \( \mathcal{Y}_s \) in a manner consistent with the definition of \( (X, \mathcal{X}, m_X) \), and (iii) \( \text{Pl}_Y(\mathcal{Y}_s) \) constituting an upper bound (technically, a supremum or ‘sup’) on the probabilities that could be assigned to \( \mathcal{Y}_s \) in a manner consistent with the definition of \( (X, \mathcal{X}, m_X) \).

Although all the CCBFs and CCPFs in this presentation will be calculated with the Monte Carlo procedure described in Table 1, it is worthwhile to illustrate a direct construction of the CCBF and CCPF in Fig. 6 to build intuition with respect to the nature of belief and plausibility and also to provide motivation for the use of Monte Carlo procedures to estimate belief and plausibility. Specifically, \( \text{Bel}(\mathcal{Y}_s) \) and \( \text{Pl}(\mathcal{Y}_s) \) are determined for \( v = 0.8 \) and 1.5. The sets \( X_{0.8} \) and \( X_{1.5} \) corresponding to \( \mathcal{Y}_{0.8} \) and \( \mathcal{Y}_{1.5} \) contain the points to the right of the contour lines for \( y = 0.8 \) and 1.5 (see Fig. 5). For \( X_{0.8}, X_k \subset X_{0.8} \) for \( k = 1, 2, 5, 6, 9, 10, \) and \( X_k \cap X_{0.8} \neq \phi \) for \( k = 1, 2, \ldots , 12; \) for \( X_{1.5}, X_k \subset X_{1.5} \) does not hold for any \( k, \) and \( X_k \cap X_{1.5} \neq \phi \) for \( k = 1, 2, 3, 4. \) Thus, the sets \( \mathcal{Y}_{0.8} \) and \( \mathcal{Y}_{1.5} \) in Eqs. (3.19) and (3.21) are given by

\[
\mathcal{Y}_{0.8} = \{1, 2, 5, 6, 9, 10\}, \quad \mathcal{Y}_{1.5} = \{1, 2, \ldots , 12\},
\]

(6.37)

and the corresponding plausibilities and beliefs are given by

\[
\text{Bel}_Y(\mathcal{Y}_{0.8}) = \sum_{k \in \mathcal{Y}_{0.8}} m_X(X_k) = 6(1/12) = 1/2
\]

(6.39)

\[
\text{Pl}_Y(\mathcal{Y}_{0.8}) = \sum_{k \in \mathcal{Y}_{0.8}} m_X(X_k) = 12(1/12) = 1.0
\]

(6.40)

\[
\text{Bel}_Y(\mathcal{Y}_{1.5}) = \sum_{k \in \mathcal{Y}_{1.5}} m_X(X_k) = 0
\]

(6.41)

as indicated in Eqs. (3.23) and (3.25). The preceding values for \( \text{Bel}_Y(\mathcal{Y}_{0.8}), \text{Pl}_Y(\mathcal{Y}_{0.8}), \text{Bel}_Y(\mathcal{Y}_{1.5}) \) and \( \text{Pl}_Y(\mathcal{Y}_{1.5}) \) on the corresponding CCBF and CCPF can be seen in Fig. 6.

Values of \( \mathcal{Y}_{0.8}, \mathcal{Y}_{1.5} \), \( \text{Bel}_Y(\mathcal{Y}_s) \) and \( \text{Pl}_Y(\mathcal{Y}_s) \) for all \( v \in \mathcal{Y} \) are summarized in Table 6. As indicated in conjunction with Eqs. (3.9) and (3.11), the points \([v, \text{Bel}_Y(\mathcal{Y}_s)]\) and \([v, \text{Pl}_Y(\mathcal{Y}_s)]\) for \( v \in \mathcal{Y} \) define the CCBF and CCPF associated with the evidence space \( (\mathcal{Y}, v, m_Y) \). In practice, determination of CCBFs and CCPFs through the direct compilation of results of the form in Table 6 is, at best, tedious and, often, computationally intractable. For the preceding reasons, the Monte Carlo approach to the estimation of CCBFs and CCPFs described in Table 1 has considerable appeal.

The construction of CCBFs, CCBFs and CCPFs for Problem 3b has been presented in considerable detail. The CCBFs, CCBFs and CCPFs that result for Problems 1a, 2a, 2b, 2c, 3a and 3c can be obtained with similar assumptions and procedures as outlined at the beginning of this section and are presented without additional discussion of their construction (Fig. 7).

The determination of CCNFs and CCPoFs is now considered. As for probability theory and evidence theory, Problem 3b is used for illustration. The solution of this problem with possibility theory involves possibility spaces \((\mathcal{X}, r_X), (\mathcal{X}, r_X), (\mathcal{X}, r_X)\) and \((\mathcal{Y}, r_Y)\). These spaces are summarized in Table 7; further, a graphical representation for \( r_X \) is provided in Fig. 8.

Problem 3b is simple enough that the associated CCNF and CCPoF can be obtained by mapping the level curves of \( y = f(x) \) (Fig. 4b) onto the plot for \( r_X \) in Fig. 8. Then, for \( \mathcal{Y}_s \) and \( X_k \), as defined in conjunction with Eq. (6.33), the values for \( \text{Pos}_Y(\mathcal{Y}_s) = \text{Pos}_X(\mathcal{Y}_s) \) and \( \text{Pos}_Y(\mathcal{Y}_s) = \text{Pos}_X(\mathcal{X}^c) \) can be read directly from the indicated figure, which is all that is needed to determine the CCNF and CCPoF for \( y \). For example,

\[
\text{Pos}_Y(\mathcal{Y}_{0.8}) = \text{Pos}_X(\mathcal{X}_{0.8})
\]

(6.43)

\[
\text{Pos}_Y(\mathcal{Y}_{0.8}) = \max \{1/4, 1/3, 1/2, 2/3, 3/4, 1\} = 1
\]

(6.44)

\[
\text{Pos}_Y(\mathcal{Y}_{1.5}) = \max \{1/4, 1/3, 1/2\} = 1/2
\]

(6.45)

\[
\text{Pos}_Y(\mathcal{Y}_{1.5}) = \text{Pos}_X(\mathcal{X}_{1.5}) = \max \{1/4, 1/3\} = 1/3
\]

(6.46)

\[
\text{Pos}_Y(\mathcal{Y}_{1.5}) = \text{Pos}_X(\mathcal{Y}_{1.5})
\]

(6.47)

\[
\text{Pos}_Y(\mathcal{Y}_{1.5}) = \max \{1/4, 1/3, 1/2, 2/3, 3/4, 1\} = 1
\]

(6.48)

as can be readily determined from the level curves for \( y = 0.8 \) and \( y = 1.5 \) in Fig. 8. The resultant CCNF and CCPoF are shown in Fig. 9.
Table 6

Summary of sets $ICC\tilde{B}_F$ and $ICC\tilde{P}_F$ (see Eqs. (3.19) and (3.20)) and corresponding beliefs and plausibilities $Bel_Y(\mathcal{U})$ and $Pl_Y(\mathcal{U})$ (see Eqs. (3.23) and (3.25)) that define the CCBF and CCPF associated with the evidence space $(\mathcal{X}, Y, m_Y)$ for Problem 3b

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$ICC\tilde{B}_F$</th>
<th>$ICC\tilde{P}_F$</th>
<th>$Bel_Y(\mathcal{U})$</th>
<th>$Pl_Y(\mathcal{U})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 0.69220$</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.69220, 0.70711)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.83333</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.70711, 0.75612)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.75000</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.75612, 0.77460)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.58333</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.77460, 0.89751)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.50000</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.89751, 0.94868)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.33333</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.94868, 0.95620)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.25000</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.95620, 0.95635)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.16667</td>
<td>1.00000</td>
</tr>
<tr>
<td>(0.95635, 1.04881)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.08333</td>
<td>1.00000</td>
</tr>
<tr>
<td>(1.04881, 1.11560)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.91667</td>
</tr>
<tr>
<td>(1.11560, 1.17049)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.83333</td>
</tr>
<tr>
<td>(1.17049, 1.20160)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.75000</td>
</tr>
<tr>
<td>(1.20160, 1.22371)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.66667</td>
</tr>
<tr>
<td>(1.22371, 1.26558)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.58333</td>
</tr>
<tr>
<td>(1.26558, 1.32578)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.50000</td>
</tr>
<tr>
<td>(1.32578, 1.32820)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.41667</td>
</tr>
<tr>
<td>(1.32820, 1.44982)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.33333</td>
</tr>
<tr>
<td>(1.44982, 1.60000)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.25000</td>
</tr>
<tr>
<td>(1.60000, 1.70000)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.16667</td>
</tr>
<tr>
<td>(1.70000, 1.80000)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.08333</td>
</tr>
<tr>
<td>(1.80000, 2.00000)</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

Fig. 7. Estimated CCBFs, CCDFs and CCPFs for $y = f(a, b) = (a + b)^a$ in Problems 1, 2a, 2b, 2c, 3a and 3c obtained with random samples of size $n_S = 10,000$. 
In practice, the function $f$ and the possibility space $(\mathcal{X}, r_X)$ are likely to be too complex to permit a direct construction of CCNFs and CCPoFs as just indicated. Rather, a numerical approach such as the Monte Carlo procedure indicated in Table 2 will be required. Indeed, even for a problem as simple as the one under consideration (i.e. Problem 3b), a direct construction of CCNFs and CCPoFs is tedious. For this reason, the possibility theory solution for Problem 3b in Fig. 9 was actually obtained with the Monte Carlo procedure in Table 2, the probability distribution $(\mathcal{X}, X, p_X)$ defined in Table 4, and the same sample size of $nS = 10,000$ used in the determination of the CCDF, CCBF and CCPF in Fig. 6.

The construction of CCNFs and CCPoFs for Problem 3b has been presented in detail. The CCNFs and CCPoFs result for Problems 1, 2a, 2b, 2c, 3a and 3c (Fig. 10) can be obtained with similar assumptions and procedures as outlined at the beginning of this section. The CCNFs and CCPoFs for Problems 2c, 3a and 3b are presented in Fig. 10. The CCNFs and CCPoFs for Problems 1, 2a and 2b are identical to the corresponding CCBFs and CCPFs for Problems 1, 2a and 2b in Fig. 7 and therefore are not presented. In the details of implementation, the solution procedures for Problems 1, 2a, 2b and 3a are effectively identical to the procedures for Problem 3b. The solution procedures for Problems 2c and 3c differ in that some of the given information is inconsistent (i.e. the specified uncertainty intervals have no common intersection; see Problems 2c and 3c in Table 3). This inconsistency necessitates the use of the scaling techniques in Eqs. (6.23) and (6.24) in the definition of $(\mathcal{B}, r_B)$ for Problem 2c and in the definition of both $(\mathcal{A}, r_A)$ and $(\mathcal{B}, r_B)$ for Problem 3c. In particular, the additive scaling in Eq. (6.23) results in the evidence space $(\mathcal{A}, r_{A,add})$ and $(\mathcal{B}, r_{B,add})$ for Problem 3c, and the multiplicative scaling in Eq. (6.24) results in the space $(\mathcal{B}, r_{B,mult})$ for Problem 2c and in the spaces $(\mathcal{A}, r_{A,mult})$ and $(\mathcal{B}, r_{B,mult})$ for Problem 3c. In turn, this results in two definitions of $(\mathcal{A}, r_A)$ and $(\mathcal{B}, r_B)$ for Problems 2c and 3c, and thus two pairs of CCNFs and CCPoFs for these problems as shown in Fig. 10.

7. More complex problem involving epistemic uncertainty

The problems considered in Section 6 are very simple in their algebraic structure. Although it would be tedious, all of these problems could be solved for evidence theory and possibility theory by use of the level curves for $y = f(x)$ (Fig. 4b) and corresponding plots of the sets in $\mathcal{X}$ with nonzero BPA's for evidence theory (e.g. Fig. 5) and the distribution function $r_X$ defined on $\mathcal{X}$ for possibility theory (e.g. Fig. 8). As $f$ and the spaces $(\mathcal{X}, X, p_X)$ and $(\mathcal{X}, r_X)$ become more complex, this approach becomes impractical.

For purposes of illustration, the propagation of uncertainty through a more complex function is considered in this
section. Specifically,

\[ f(U, V) = U + V + UV + U^2 + V^2 + g(V) \cos[3\pi(U - 1.25)] \]

with

\[ g(V) = \min\{\max\{1/(V - 11/43) + 1/(V - 22/23) + 1/(V - 33/44), -10\}, 10\} \]

and the variables \( U \) and \( V \) are defined on \([1.0, 1.5]\) and \([0.0, 1.0]\), respectively, with specified intervals of equal likelihood, or credibility (Fig. 11). In particular, three cases are considered: (i) five intervals for \( U \) and \( V \) as indicated in Fig. 11, (ii) 10 intervals for \( U \) and \( V \) obtained by dividing each interval indicated in Fig. 11 in half (e.g. \([0.0, 0.4]\) for \( V \) is divided in to \([0.0, 0.2]\) and \([0.2, 0.4]\)), and (iii) 20 intervals for \( U \) and \( V \) obtained by dividing each interval indicated in Fig. 11 into quarters (e.g. \([0.0, 0.4]\) for \( V \) is divided into \([0.0, 0.1]\), \([0.1, 0.2]\), \([0.2, 0.3]\) and \([0.3, 0.4]\)). The half-open intervals indicated for cases (ii) and (iii) are assumed to avoid technical problems in the numerical implementation of evidence theory and possibility theory when specified intervals intersect in a single point. Such intersections result in single points being assigned distinct BPAs and possibility

![Fig. 10. Estimated CCNFs, CCDFs and CCPoFs for \( y = f(a, b) = (a + b)^{a}\) in Problems 2c, 3a and 3c obtained with random samples of size \( n_S = 10,000 \) and either additive scaling (see Eq. (6.23)) or multiplicative scaling (see Eq. (6.24)) used in the generation of the evidence space \( (\mathcal{B}, r_B) \) for Problem 2c and in the generation of the evidence spaces \( (\mathcal{A}, r_A) \) and \( (\mathcal{B}, r_B) \) for Problem 3c.](image)

![Fig. 11. Function \( y = f(U, V) \) defined in Eq. (7.1) and associated uncertainty characterizations for \( U \) and \( V \).](image)
distribution values and in turn result in a requirement for the Monte Carlo procedures used to estimate CCPFs, CCBFs, CCPOFs and CCNFs to employ sampling distributions that assign nonzero probabilities to these points.

As discussed in Section 6 with \( x = [U, V] \) instead of \( x = [a, b] \), the preceding specifications lead to probability spaces \((\Omega, \mathcal{U}, P_U), (\Omega, \mathcal{V}, P_V), (\Omega, \mathcal{X}, P_X)\), evidence spaces \((\Omega, \mathcal{U}, m_U), (\Omega, \mathcal{V}, m_V), (\Omega, \mathcal{X}, m_X)\), and possibility spaces \((\Omega, r_{U,\text{add}}), (\Omega, r_{V,\text{add}}), (\Omega, r_{X,\text{add}}), (\Omega, r_{U,\text{mult}}), (\Omega, r_{V,\text{mult}}), (\Omega, r_{X,\text{mult}})\) for the three cases, with the subscripts add and mult used in the designation of the possibility spaces to indicate whether the necessary scaling was performed with the additive procedure in Eq. (6.23) or the multiplicative procedure in Eq. (6.24). In particular, the indicated spaces are defined in exactly the same manner as the corresponding spaces are defined for Problem 3b in Section 6. Given the manner in which the intervals used for \( U \) and \( V \) in case (i) are subdivided to produce cases (ii) and (iii), the indicated probability spaces are the same for all three cases; scaling is only required in the definition of possibility spaces for \( U \) and \( V \) for cases (ii) and (iii), and multiplicative scaling for \( U \) and \( V \) for cases (ii) and (iii) produces the same possibility spaces as in case (i).

As is evident from the structure of \( y = f(x) \), this problem is too complex to determine CCBFs, CCPFs, CCNFs and CCPOFs for \( y \) by inspection of the appropriate plots. Further, the use of formal integration procedures to obtain the probabilities that define the CCDF for \( y \) is not straightforward. However, the preceding results are relatively easy to obtain with Monte Carlo procedures.

Estimates of CCBFs, CCDFs and CCPFs for the three cases obtained with a sample of size \( nS = 10,000 \) are given in Fig. 12. The CCBFs and CCPFs are drawing closer to the CCDF as the amount of detail in the definition of \((\Omega, \mathcal{X}, m_X)\) is increased (i.e. as \( U \) and \( V \) are divided into more intervals with assigned BPAs, and as a result, the amount of resolution in the specification of \((\Omega, \mathcal{U}, m_U)\) and \((\Omega, \mathcal{V}, m_V)\) and hence in the specification of \((\Omega, \mathcal{X}, m_X)\) increases). In particular, \( X \) has 25, 100 and 400 elements for cases (i), (ii) and (iii), respectively. Experimentation with different sample sizes indicates that the results for \( nS = 10,000 \) in Fig. 12 can be regarded as fully converged estimates of the CCBFs, CCDFs and CCPFs for cases (i), (ii) and (iii). Thus, the Monte Carlo procedure in Table 1 can be used to estimate the CCBF and CCPF for a fairly complicated function (e.g. \( f \) as defined in Eq. (7.1)) and
illustrated in Fig. 11) as long as the evaluation of this function is not too expensive computationally.

Increasing the number of elements in the set $\mathcal{X}$ associated with an evidence space $(X, X_m, m_X)$ tends to increase the sample size required to estimate CCBFs and CCPFs for a function defined on $X$. As an example, estimates of the CCBF and CCPF for case (iii) are shown in Fig. 13 for sample sizes of $nS = 100$, 250 and 500, respectively. The results for $nS = 100$ and 250 are clearly inadequate (i.e. the estimated CCBFs and CCPFs emerge from the ordinate at values less than one, which indicates that all elements of $\%$ have not been intersected). The sample of size 500 results in the intersection of all elements of $\%$ but does not result in estimates of the CCBF and CCPF that are fully converged (i.e. to the results for a sample size 10,000 in Fig. 12c, which can be regarded as the fully converged result).

Estimates of CCNFs, CCDFs and CCPoFs for the three cases obtained with a sample of size $nS = 10,000$ and additive scaling of the possibility distribution functions for $U$ and $V$ for cases (ii) and (iii) are given in Fig. 14. No scaling is required for case (i), and multiplicative scaling for cases (ii) and (iii) produces CCBFs and CCPoFs that are the same as those in Fig. 14a; this is not an inherent property of multiplicative scaling but rather results from the manner in which possibility distribution functions are being defined and the even division of the intervals associated with case (i) to produce cases (ii) and (iii). Similarly to the evidence theory results illustrated in Fig. 13, increasing the complexity of the distribution function associated with a possibility space increases the size of the random sample required to obtain converged estimates for CCNFs and CCPoFs. As for the evidence theory results in Fig. 12, this example illustrates that the Monte Carlo procedures in Table 2 can be used to estimate the CCNF and CCPoF for a fairly complicated function as long as the evaluation of this function is not too expensive computationally.

In real problems, computational expense is a constraint that cannot be avoided. A recent uncertainty and sensitivity analysis of a two-phase fluid flow problem serves as an example [79–82]. This analysis involved 31 uncertain analysis inputs, several variants of the analysis problem, and a large numerical model based on a system of nonlinear partial differential equations that required 1–4 h of CPU time per model evaluation. Without some type of additional refinement, the Monte Carlo procedures in Tables 1 and 2 to obtain evidence theory and possibility theory representations of uncertainty would be prohibitively expensive from a computational perspective for problems of this type.

8. Algebraic problem set: epistemic and aleatory uncertainty

This section involves the same simple algebraic model considered in Section 6 (see Eq. (6.1)). However, while $a$ is still assumed to be a fixed quantity with epistemic uncertainty with respect to its true value, $b$ is now assumed to be a randomly varying (i.e. aleatoric) quantity with epistemic uncertainty with respect to the distribution that

| Problem 4: |
| $A = [0.1, 1.0], M = [0.1, 1.0], S = [0.1, 0.5]$ |
| Problem 5a: |
| $A_1 = [0.5, 0.7], A_2 = [0.3, 0.8], A_3 = [0.1, 1.0]$ |
| $M_1 = [0.6, 0.8], M_2 = [0.2, 0.9], M_3 = [0.0, 1.0]$ |
| $S_1 = [0.3, 0.4], S_2 = [0.2, 0.45], S_3 = [0.1, 0.5]$ |
| Problem 5b: |
| $A_1 = [0.5, 1.0], A_2 = [0.2, 0.7], A_3 = [0.1, 0.6]$ |
| $M_1 = [0.6, 0.9], M_2 = [0.1, 1.0], M_3 = [0.0, 0.1]$ |
| $S_1 = [0.3, 0.45], S_2 = [0.15, 0.35], S_3 = [0.1, 0.5]$ |
| Problem 5c: |
| $A_1 = [0.8, 1.0], A_2 = [0.5, 0.7], A_3 = [0.1, 0.4]$ |
| $M_1 = [0.6, 0.8], M_2 = [0.1, 0.4], M_3 = [0.0, 1.0]$ |
| $S_1 = [0.4, 0.5], S_2 = [0.25, 0.35], S_3 = [0.1, 0.2]$ |
| Problem 6: |
| $A = [0.1, 1.0], \mu = 0.5, \sigma = 0.5$ |
Table 9
Summary of information related to probability, evidence and possibility spaces in Problem 5a

Part 1: Preliminary definitions

\[ \mathcal{A}_1 = \{a : 0.5 \leq a \leq 0.7\}, \mathcal{A}_2 = \{a : 0.3 \leq a \leq 0.8\}, \mathcal{A}_3 = \{a : 0.1 \leq a \leq 1.0\} \]

\[ \mathcal{A}_i = \{a : 0.1 \leq a \leq 1.0\} \]

\[ \mathcal{X}_i = \{a : 0.6 \leq a \leq 0.8\}, \mathcal{X}_2 = \{a : 0.2 \leq a \leq 0.9\}, \mathcal{X}_3 = \{a : 0.0 \leq a \leq 1.0\} \]

\[ \mathcal{X}_1 = \{\sigma : 0.3 \leq \sigma \leq 0.4\}, \mathcal{X}_2 = \{\sigma : 0.2 \leq \sigma \leq 0.45\}, \mathcal{X}_3 = \{\sigma : 0.1 \leq \sigma \leq 0.5\} \]

\[ \mathcal{B}_1 = \{b : \mu = 0.6, \mu \leq 0.8\}, \mathcal{B}_2 = \{b : \mu = 0.2, \mu \leq 0.9\}, \mathcal{B}_3 = \{b : \mu = 0.0, \mu \leq 1.0\} \]

\[ \mathcal{S}_i = \{\sigma : 0.3 \leq \sigma \leq 0.4\}, \mathcal{S}_2 = \{\sigma : 0.2 \leq \sigma \leq 0.45\}, \mathcal{S}_3 = \{\sigma : 0.1 \leq \sigma \leq 0.5\} \]

\[ \mathcal{Y}_i = \{\sigma : 0.3 \leq \sigma \leq 0.4\}, \mathcal{Y}_2 = \{\sigma : 0.2 \leq \sigma \leq 0.45\}, \mathcal{Y}_3 = \{\sigma : 0.1 \leq \sigma \leq 0.5\} \]

\[ \mathcal{D}_i = \{d : \mu = 0.6, \mu \leq 0.8\}, \mathcal{D}_2 = \{d : \mu = 0.2, \mu \leq 0.9\}, \mathcal{D}_3 = \{d : \mu = 0.0, \mu \leq 1.0\} \]

\[ \mathcal{Y}_1 = \{\sigma : 0.3 \leq \sigma \leq 0.4\}, \mathcal{Y}_2 = \{\sigma : 0.2 \leq \sigma \leq 0.45\}, \mathcal{Y}_3 = \{\sigma : 0.1 \leq \sigma \leq 0.5\} \]

\[ \mathcal{A}_i = \{a : 0.1 \leq a \leq 1.0\} \]

Part 2: Probability spaces \( (\mathcal{A}, \mathcal{B}, \mathcal{P}_\mathcal{A}) \), \( (\mathcal{A}, \mathcal{B}, \mathcal{P}_\mathcal{B}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{Y}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{Y}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}) \)

Part 3: Evidence spaces \( (\mathcal{X}, \mathcal{Z}, \mathcal{X}_3, \mathcal{Y}, \mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{P}_\mathcal{Y}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{X}_3) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{X}_3, \mathcal{Y}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{P}_\mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}) \)

Part 4: Possibility spaces \( (\mathcal{X}, \mathcal{Z}, \mathcal{X}_3) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}) \), \( (\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}, \mathcal{P}_\mathcal{X}) \)

characterizes this variance. In particular, \( b \) is assumed to follow a lognormal distribution with an uncertain mean \( \mu \) and an uncertain standard deviation \( \sigma \) (i.e. \( b \) has mean \( \mu \) and standard deviation \( \sigma \), with both of these parameters being uncertain). Thus, in the problems considered in this section (i.e. Problems 4, 5a, 5b, 5c, 6), there is epistemic uncertainty in the values for \( a, \mu, \) and \( \sigma \) (Table 8), and aleatory uncertainty in the values for \( b \) characterized by a lognormal distribution defined by \( \mu \) and \( \sigma \). The maintenance of a conceptual and computational distinction between aleatory uncertainty (i.e. random variation in system behavior) and epistemic uncertainty (i.e. state of knowledge uncertainty with respect to the appropriate values to use for quantities that are assumed to have fixed, but poorly known, values in the context of a particular analysis) is common in many analyses of complex systems [83–90].

In each problem summarized in Table 8, \( nA \) and \( nB \) independent sources provide uncertainty ranges for \( a, \mu, \) and \( \sigma \) (i.e. \( nA = nB = 1 \) for Problems 4 and 6; \( nA = nB = 3 \) for Problems 5a, 5b, 5c). The development of probability, evidence and possibility spaces for the problems in Table 8 is analogous to the development of these spaces for the problems in Table 3 with one variation. This variation is that each source for the distribution characterizing aleatory
uncertainty in \( b \) gives a range for \( \mu \) and also a range for \( \sigma \). Thus, each source is actually specifying a set that contains possible values for the vector \( \mathbf{b} = [\mu, \sigma] \). As a result, the vector \( \mathbf{x} \) of variables with specified epistemic uncertainty is

\[
x = [a, b] = [a, \mu, \sigma],
\]

and probability, evidence and possibility spaces are developed for \( a \) and \( b \) rather than for \( a, \mu \) and \( \sigma \) individually. The definition of the preceding spaces for Problem 5a is summarized in Table 9; the development of these spaces for the other problems in Table 8 is similar [91].

As indicated in Table 9, each element \( \mathbf{x} = [a, \mu, \sigma] \) of \( \mathcal{X} \) gives rise to a probability space \( (\Theta(x), \gamma(x), p_{Y|x}(x)) \) characterizing aleatory uncertainty in \( y \). Thus, at the most basic level, the epistemic uncertainty associated with the problems in Table 8 gives rise to many possible probability spaces for values of \( y \) that arise from aleatory uncertainty (i.e. the set \( \mathcal{P} \) in Table 9). In practice, because probability spaces are too complex to be considered graphically or numerically as single, distinct entities, various summary measures that can be derived from the definition of a probability space are often used as surrogates for the space. Such measures include means (i.e. the set \( \mathcal{M} \) in Table 9), exceedance probabilities which in essence define CCDFs (i.e. the set \( \mathcal{P}(y) \) in Table 9), and mean–standard deviation pairs for individual probability spaces (i.e. the set \( \mathcal{M}_{PS} \) in Table 9). In turn, whatever uncertainty formalism is defined on \( \mathcal{X} \) induces a corresponding formalism on \( \mathcal{M}, \mathcal{P}(y), \mathcal{M}_{PS} \) and any other sets of summary measures based on the probability spaces in \( \mathcal{P} \) that are selected for consideration.

For illustration, a subset of the CCDFs that derive from the probability spaces \( (\Theta(x), \gamma(x), p_{Y|x}(x)) \) in Problem 5a is shown in Fig. 15. In particular, the 50 CCDFs in Fig. 15 were produced by first generating a random sample \( x_i \), \( i = 1, 2, \ldots, nX = 50 \), from \( \mathcal{X} \) in consistency with the definition of the probability space \( (\Theta, \gamma, p_Y) \). This produced the probability spaces \( (\Theta(x_i), \gamma(x_i), p_{Y|x}(x_i)) \), \( i = 1, 2, \ldots, nX = 50 \). Each of the CCDFs in Fig. 15 derives from one of the indicated probability spaces and, in concept, could be produced by a formal integration process. However, it is easier to use a Monte Carlo procedure. In particular, a random sample of size 200 was generated from \( \Theta(x_i) \) for each probability space and used to construct the corresponding CCDF in Fig. 15. With respect to conceptual interpretation, the multiple CCDFs in Fig. 15 arise from epistemic uncertainty associated with \( \mathbf{x} = [a, \mu, \sigma] \), and each individual CCDF arises from aleatory uncertainty in possible values for \( y = f(a, b) \) that derives from the distribution associated with \( b \). For all practical purposes, the CCDFs in Fig. 15 effectively define the corresponding probability spaces \( (\Theta(x_i), \gamma(x_i), p_{Y|x}(x_i)) \) for \( y \). Further, each CCDF in Fig. 15 can be reduced to a mean \( m(x_i) \), a mean–standard deviation pair \( [m(x_i), s(x_i)] \), and for each \( y \) on the abscissa, an exceedance probability \( p_y(x) \), with these quantities forming subsets of \( \mathcal{M}, \mathcal{M}_{PS} \) and \( \mathcal{P}(y) \), respectively.

The characterizations of subsets of \( \mathcal{X} \) with the probability, evidence and possibility spaces indicated in Table 9 can be used to obtain corresponding characterizations of the uncertainty associated with \( \mathcal{P}, \mathcal{M}, \mathcal{M}_{PS} \) and \( \mathcal{P}(y) \). The set \( \mathcal{M} \) of means is used for illustration. As described in Table 9, probability, evidence and possibility spaces \( (\mathcal{M}, \mathcal{M}_{PS}, \mathcal{P}(y)) \) and \( (\mathcal{M}, \mathcal{M}_{PS}, \mathcal{P}(y)) \) can be defined for the means contained in \( \mathcal{M} \). In turn, the Monte Carlo procedures as indicated in conjunction with Eq. (2.12) and Tables 1 and 2 can be used to estimate CCDFs, CCBFs, CCPFs, CCNFs and CCPoFs (Fig. 16).

The CCDF in Fig. 16 derives from the probability space \( (\mathcal{M}, \mathcal{M}_{PS}, \mathcal{P}(y)) \) in Table 9. Similarly, the CCBF and CCPF derive from the evidence space \( (\mathcal{M}, \mathcal{M}_{PS}, \mathcal{P}(y)) \) in Table 9, and the CCNF and CCPoF derive from the possibility space \( (\mathcal{M}, \mathcal{M}_{PS}, \mathcal{P}(y)) \) in Table 9. The same sampled results (see caption to Fig. 16) used in the generation of the CCDF in Fig. 16 are also used in the generation of the CCBF, CCPF, CCNF and CCPoF in that figure. In particular, no attempt is made to determine \( \mathcal{M} \) and \( m_M \) for use in construction of the CCBF and CCPF for \( m \), and similarly, no attempt is made to determine \( r_M \) for use in construction of the CCNF and CCPoF for \( m \). Rather, the need to determine \( \mathcal{M} \), \( m_M \) and \( r_M \) is avoided by use of the Monte Carlo procedure.

For perspective, results for Problems 4, 5b, 5c and 6 corresponding to those in Figs. 15 and 16 for Problem 5a are
presented in Fig. 17. The underlying probability, evidence and possibility spaces for these problems are defined in the same manner as described in Table 9 for the corresponding spaces for Problem 5a with one exception. This exception occurs for Problem 5c, where the initially defined possibility distribution functions \( r_A \) and \( r_B \) equal 1/3 for all elements of \( A \) and \( B \), respectively, and thus are scaled to equal 1. Because \( r_A \) and \( r_B \) are constant, both multiplicative and additive scaling produce the same result (see Eqs. (6.23) and (6.24)).

To this point, only the probability, evidence and possibility spaces associated with the set \( M \) of means over aleatory uncertainty have been considered. As indicated in Table 9, it is possible to define probability, evidence and possibility spaces over many other quantities for the problems under consideration in this section, including probability spaces, exceedance probabilities, and vectors of mean–standard deviation pairs (i.e. the sets \( \mathcal{P}, \mathcal{P}(y) \) and \( \mathcal{M}, \mathcal{M}(y) \) in Table 9).

As one additional example, probability, evidence and possibility results are presented for \( \mathcal{P}(y) \) in a modified version of Problem 5a. In particular, the values for \( a, \mu \) and \( \sigma \) given in Table 8 for Problem 5a are each increased by 1; no other modifications to Problem 5a or the resultant quantities defined in Table 9 are made. The effect of these modifications is to produce CCDFs that are more spread out than those in Figs. 16 and 17 and thus provide a better visual example of the elements of the set \( \mathcal{P}(y) \).

Examples of the CCDFs that derive from the probability spaces \( (\mathcal{P}(x), \mathcal{Y}(x), p_{Y|x}) \) that result for individual elements \( x \) of \( \mathcal{X} \) are shown in Fig. 18a. For each \( y \), \( \mathcal{P}(y) \) is the set of all exceedance probabilities for \( y \) that derive from possible values for \( x \). For example, the exceedance probabilities associated with the CCDFs intersected by the vertical line labeled \( y = 10^3 \) in Fig. 18a constitute a subset of \( \mathcal{P}(10^3) \). As indicated in Table 9, there exist probability, evidence and possibility spaces associated with these exceedance probabilities (i.e. \( \mathcal{P}(y), \mathcal{P}(y), p_{\mathcal{P}(y)} \), \( \mathcal{P}(y), \mathcal{P}(y), m_{\mathcal{P}(y)} \) and \( \mathcal{P}(y), r_{\mathcal{P}(y)} \), respectively, for \( y = 10^3 \)). In turn, these spaces have associated CCDFs, CCBFs, CCPFs, CCNFs and CCPoFs that provide summaries of the uncertainty in the exceedance probability for \( y = 10^3 \) (Fig. 18b and c).

The importance of maintaining a distinction between aleatory and epistemic uncertainty dominates the design and computational cost of many analyses for complex systems. For example, the US Nuclear Regulatory Commission’s study of the risk from commercial nuclear power plants (often called the NUREG-1150 analysis after its report number) involved a conceptual and computational separation of aleatory and epistemic uncertainty [92–98]. The NUREG-1150 analysis used probability to represent both aleatory and epistemic uncertainty, with the integrations associated with aleatory uncertainty implemented by importance sampling procedures defined by large event and fault trees and integrations associated with epistemic uncertainty implemented by Latin hypercube sampling. The NUREG-1150 analysis was carried out in response to reviews of an earlier analysis of reactor safety (often called the Reactor Safety Study or the WASH-1400 analysis after its report number) which praised the characterizations of aleatory uncertainty in the analysis but criticized the limited characterization of epistemic uncertainty [99,100]. As another example, analyses in support of the Compliance
Fig. 17. Multiple aleatoric distributions of $y = f(a, b) = (a + b)^b$ resulting from epistemic uncertainty in $x = [a, \mu, \sigma]$ in Problems 4, 5b, 5c and 6 (left column) and corresponding estimated CCDFs, CCBFs, CCPFs, CCNFs and CCPs for set $M$ of means over aleatory uncertainty (right two columns) with results in left column and right two columns generated with same procedures and sample sizes are used in the generation of corresponding results in Figs. 15 and 16, respectively.
Certification Application for the Waste Isolation Pilot Plant (WIPP) maintained a separation between aleatory and epistemic uncertainty, with this separation mandated in regulations for that facility promulgated by the US Environmental Protection Agency [101–103]. In the analysis for WIPP, as in the NUREG-1150 analysis, probability was used to represent both aleatory and epistemic uncertainty, with the integrations associated with aleatory uncertainty and epistemic uncertainty being carried out with simple random sampling and Latin hypercube sampling, respectively.

As illustrated in this section, evidence theory and possibility theory can also be used in analyses that maintain a separation between aleatory and epistemic uncertainty. In particular, the example problems of this section use probability to characterize aleatory uncertainty and then provide alternative representations of epistemic uncertainty based on probability theory, evidence theory and possibility theory. However, the naïve Monte Carlo methods used to obtain evidence theory and possibility theory results for the example problems in this section are unlikely to be computationally practicable in a real analysis due to the large number of model evaluations that would be required. The NUREG-1150 analyses involved 100–200 uncertain analysis inputs for each of five different nuclear power plants, and the WIPP analysis involved 56 uncertain analysis inputs and a sequence of complex models. It is unlikely that either of these analyses could have been carried out for evidence theory or possibility theory with Monte Carlo techniques as illustrated in this section. Rather, some type of more sophisticated implementation procedure would be needed. Indeed, carrying out these analyses for probabilistic characterizations of uncertainty required importance sampling implemented through event trees and fault trees, stratified sampling implemented through Latin hypercube sampling, efficient use and reuse of results obtained from computationally demanding models, appropriate aggregation of results at various ‘pinch points’ in the analyses, and procedures for eliminating ‘unimportant’ results to keep the analyses at a manageable scale.


The mass-spring-damper (MSD) problem is summarized in Table 10, with the steady-state magnification factor $D_s$ being the quantity selected for analysis. In the problem description summarized in Table 10, and given in more detail in Ref. [51], the spring mass $m$ has a completely specified triangular distribution with a minimum of $m_{\text{min}} = 10$, a mode of $m_{\text{mod}} = 11$, and a maximum of $m_{\text{max}} = 12$; the spring constant $k$ has a triangular distribution with uncertainty ranges specified for the minimum, $k_{\text{min}}$, mode, $k_{\text{mod}}$, and maximum, $k_{\text{max}}$, of this distribution by three independent sources; the damping coefficient $c$ is uncertain with uncertainty ranges specified by three independent sources; the amplitude $Y$ of the forcing function is uncertain with a single specified range; and the frequency $\omega$ of the forcing function was a triangular distribution with single uncertainty ranges specified for a minimum, $\omega_{\text{min}}$, mode, $\omega_{\text{mod}}$, and maximum, $\omega_{\text{max}}$, of this distribution.

In the following analysis, epistemic uncertainty is assumed to exist with respect to the appropriate values of the elements of the vector

$$x = [k_{\text{min}}, k_{\text{mod}}, k_{\text{max}}, c, Y, \omega_{\text{min}}, \omega_{\text{mod}}, \omega_{\text{max}}].$$  \hfill (9.1)

In contrast, the uncertainty in the elements of the vector

$$a = [m, k, \omega]$$  \hfill (9.2)

is assumed to be of an aleatory character. In particular, the aleatory uncertainty in $m$, $k$ and $\omega$ is characterized by
### 10. Discussion

This presentation explores the propagation of uncertainty through models of the form \( y = f(x) \). Three aspects of this propagation are considered: (i) the characterization of the uncertainty in model inputs, (ii) the actual propagation of

![Fig. 19. Multiple aleatoric distributions of \( D_s \) in mass-spring-damper (MSD) problem resulting from epistemic uncertainty in \( x \) (Frame 19a) and corresponding estimated CCDF, CCBF, CCPF, CCNF and CCPoF for \( D_s \) are shown in Fig. 19b and c. Because there is little resolution in the specification of the epistemic uncertainty associated with the elements of \( x \), there is a corresponding lack of resolution in the characterization of the uncertainty in \( D_s \) with evidence and possibility theory. There is more structure in the characterization with probability theory due to the assumption of uniform distributions over specified epistemic uncertainty ranges.](https://example.com/f19.png)
uncertainty through the model, and (iii) the representation of uncertainty in model predictions. Algebraically simple models defined in conjunction with a sequence of test problems [51] are used for illustration, and four theories/approaches to the representation of uncertainty are considered: probability theory, evidence theory, possibility theory, and interval analysis.

Each problem starts with a given specification of uncertainty for the individual model inputs (i.e. the elements of x) obtained from one or more equally credible sources of information. The development of uncertainty characterizations for analysis inputs is typically one of the most demanding and expensive parts of a real analysis. Although such uncertainty characterizations may be obtained, at least in part, from direct experimentation and observation, most real analyses require some type of expert review and assessment process to convert available information into a mathematical form that is suitable for further analysis [74, 104–115]. Thus, the example problems under consideration avoid a major part of most analyses by starting with an initial specification of uncertainty.

However, the problems do not avoid the necessity to aggregate multiple sources of information. Specifically, several of the problems involve one or more variables whose uncertainty has been characterized by more than one source (e.g. expert). Thus, these multiple uncertainty characterizations must be combined (i.e. aggregated) into a single characterization before the propagation of uncertainty through the model under consideration can be carried out. Further, this aggregation must be performed in a manner that is consistent with both the nature of the supplied uncertainty information and the particular uncertainty characterization theory/ approach in use. The aggregation of multiple sources of information has been extensively studied and a number of approaches have been proposed (e.g. see Refs. [74–77] for additional discussion and extensive references).

This presentation aggregates multiple sources of information by, in effect, assigning a weight to each possible value of a variable that derives from the number of sources that specified that value. The details of this assignment change for the different uncertainty representations (i.e. probability theory, evidence theory, possibility theory) but remain the same in philosophy: namely, that the likelihood assigned to a particular variable value should derive from the number of sources that specified that value and also that no value should be eliminated (i.e. assigned a likelihood of zero) because it was not specified by one or more of the sources. For probability theory and a particular variable, this approach was implemented by developing a piecewise uniform distribution for that variable from the information supplied by each source and then averaging these distributions with equal weight assigned to each source. Analogous procedures were used for evidence theory and possibility theory. For interval analysis, there is no structured representation of likelihood, and all specified values of a variable were considered for propagation through the model. Unlike some approaches for aggregating multiple sources of information (e.g. Dempster’s Rule, which is often associated with applications of evidence theory), the approach in use results in the retention of the variable values from all sources for propagation through the model.

The actual propagation of uncertainty through the models was carried out with Monte Carlo techniques. With this approach, a sampling distribution is defined on the set of all possible variable values; a sample is generated from these values in consistency with the definition of the sampling distribution; and then the individual sample elements are used as input to the model under consideration and the resultant model predictions are saved for later use. Any sampling distribution can be used as long as it does not exclude the possibility of specific points or regions with important properties from being included in the analysis. For example, a uniform distribution on a closed interval is not appropriate if single points in that interval are associated with results that must be included in the analysis to obtain an appropriate representation of uncertainty as such points would have a probability of zero with a uniform distribution. However, the properties of the sampling distribution can have a substantial impact on the computational requirements of the analysis (i.e. on the number of required model evaluations to obtain acceptably converged results).

The presented analyses used the distributions defined to represent uncertainty with probability theory as the sampling distributions. Thus, these distributions served a dual purpose in the analyses, being used as sampling distributions for all for approaches to the representation of uncertainty and as uncertainty representation distributions for probability theory.

The cost of the model evaluations in this presentation is inconsequential, with the result that there is no meaningful limit on the sample size that can be used in the propagation of sampled inputs. However, in real analyses, this is typically not the case, with the model under consideration often being expensive to evaluate. As a result, the number of model evaluations that can be carried out is necessarily limited. In this situation, careful thought must be given to the planning of the analysis. For example, appropriate choice of the sampling distribution can enhance the resolution of the analysis for a given sample size. Similarly, use of Latin hypercube sampling [66–68] or some other efficient sampling procedure may enhance the resolution of the analysis for a given sample size.

In this presentation, the models have been treated as “black boxes,” where inputs are supplied to a model and results are obtained without any consideration of the internal structure of the model. Of course, the models under consideration are very simple and we know every aspect of their (limited) internal structure. But, the treatment of a model as a ‘black box’ is quite appropriate in many analyses. In particular, the Monte Carlo techniques in use are very convenient because their implementation is not predicated on
specific knowledge about model structure. However, as models become more complex and hence more expensive to evaluate, it is sometimes advantageous to treat them as ‘grey boxes,’ where their structure is at least partially taken into account and, as a result, computational efficiencies are achieved by making use of particular characteristics of the model (e.g. repeated calculations, linearities with respect to input variables or intermediate results, opportunities to truncate calculations or specific calculational subpaths because of inconsequential results,...). The downside of this approach to achieving computational efficiency is that it can be very demanding of (expensive) human time and, even worse, is not always possible.

Once the model evaluations are carried out, the resultant uncertainty representations can be constructed. For interval analysis, this just consists of presenting the observed range of model predictions. For probability theory, this consists of presenting CDFs or CCDFs constructed by assigning equal weight (i.e. the reciprocal of the sample size) to each model prediction if the sampling distribution is the same as the distribution used to characterize uncertainty with probability theory. If the sampling distribution is different, then the weights must be recalculated in a manner that takes into account the properties of the sampling distribution and the distribution used to represent uncertainty [116,117]. Probabilities for additional subsets of model predictions can be estimated by mapping from these subsets back to the sampled values that gave rise to these sets. Uncertainty characterizations for evidence theory and possibility theory are obtained in a similar manner. Specifically, sets of predicted values are mapped back to the sets of sampled values that gave rise to the predicted values. Then, the sets of predicted values are assigned the same plausibilities and beliefs as the sampled values if evidence theory is under consideration and the same possibilities and necessities as the sampled values if possibility theory is under consideration. In particular, this procedure provides the basis for constructing CPFs, CBFs, CCPFs and CCBFs for model predictions when evidence theory is under consideration and CPoFs, CNFs, CCPoFs and CCNFs when possibility theory is under consideration.

At a basic level, the determination of the uncertainty in model predictions with probability theory, evidence theory and possibility theory all operate in the same manner with the presented Monte Carlo techniques. Specifically, sets of model predictions are mapped back to sets of model inputs; then, the sets of model predictions are assigned the same uncertainty values (i.e. probabilities, plausibilities, beliefs, possibilities, necessities) as the corresponding sets of sampled values. Further, the interval analysis result is the range of model predictions. Thus, results with probability theory, evidence theory, possibility theory and interval analysis can all be estimated with the same sample and resultant model predictions.

The different approaches to the representation of uncertainty under consideration (i.e. probability theory, evidence theory, possibility theory, interval analysis) certainly have the potential to give rise to different appearing results when analysis outcomes are presented as CDFs, CCDFs, CPFs, CBFs, CCPFs, CCBFs, CPoFs, CNFs, CCPoFs, CCNFs or simply as intervals. A meaningful interpretation and use of such results has to be based on an understanding of the theory that underlies their genesis rather than on their physical appearance. Unfortunately, this can sometimes be very difficult when uncertainty information is supplied by a variety of sources and it is not clear exactly what interpretation the individual sources intended for their uncertainty information. Further, even if appropriate interpretive knowledge is possessed by those performing an analysis, this knowledge may be lost as the results of the analysis are passed through policy makers, decision makers, lawyers and possibly the general public.

The problems considered in this presentation have rather diffuse initial specifications of uncertainty (i.e. values specified to be contained in a single interval or a limited number of intervals). This diffuseness produces a good example of the care that must be used in interpreting different uncertainty representations. Imposition of piecewise uniform distributions in the probabilistic representation of uncertainty results in a single CCDF for a model prediction. This single CCDF can make knowledge about uncertainty appear to be more precise than it really is unless one is willing to accept that probabilities derived from uniform distributions truly represent the given information about the potential values for the variables under consideration. This assumption is suspect as the supplied information only specified intervals of values and gave no indication of likelihood within these intervals. If the feeling is that a variable is in a particular set (i.e. an interval in the problems of this presentation) and that nothing exists to distinguish between the potential values of this variable, then the use of probability can produce results that have an appearance of exactness that is not really present. Both evidence theory and possibility theory provide a way to incorporate and display this inexactness in the specification of uncertainty in model predictions. In particular, evidence theory leads to a display of the lowest and highest probabilities that are consistent with a probabilistic interpretation of the given information. Possibility theory leads to an even more diffuse representation of uncertainty. Under certain conditions (i.e. consonant specification of uncertainty), use of evidence theory and use of possibility theory can produce identical representations of uncertainty but this is not the case in general. At the extreme is interval analysis in which a range of input values gives rise to a range of model results and nothing more (i.e. there is no uncertainty structure in the range of model results).

An important topic illustrated in this presentation is the separation of aleatory and epistemic uncertainty. As is the case in most analyses, this presentation uses probability to represent aleatory uncertainty. Further, the characterization of epistemic uncertainty in the presence of aleatory
uncertainty is illustrated with probability theory, evidence theory, possibility theory, and interval analysis. In these illustrations, epistemic uncertainty is present in both the characterization of the probability distributions that define aleatory uncertainty and in variables used in the determination of analysis results. The examples in this presentation involving aleatory and epistemic uncertainty are very simple. However, the distinction between these two sources of uncertainty and the development of a computational strategy to maintain and display this distinction is a major consideration in many large analyses (e.g. probabilistic risk assessments for nuclear power plants and other complex engineered facilities). Thus, the possible representations of epistemic uncertainty in such analyses is an area of investigation that deserves careful consideration. Important topics, and thus topics that merit further investigation, include (i) characterization and aggregation of uncertainty in analysis inputs, (ii) propagation of aleatory and epistemic uncertainty through complex and computationally demanding models, and (iii) communication of analysis results to diverse groups of interested parties, including scientific peers and review groups, project sponsors, regulators, intervenors, the press, and the general public.

The characterization and aggregation of (epistemic) uncertainty in real analyses can be a large, time-consuming and expensive undertaking. This point can be made by using one large analysis as an example. The NUREG-1150 reactor probabilistic risk assessments used a very large expert review process involving several hundred outside experts to provide probabilistic characterizations of the epistemic uncertainty in hundreds of variables used in the analysis of five nuclear power plants [118,119]. The uncertainty in each variable was independently characterized by multiple experts and then these characterizations were aggregated (i.e. averaged with equal weight given to each expert’s distribution) to produce the uncertainty characterizations propagated through the analysis. The procedures and results of this uncertainty assessment are documented in a sequence of large reports [120–124]. The challenges that had to be addressed in this analysis and would have to be addressed in any uncertainty characterization process based on expert review include (i) development and implementation of procedures for the selection of suitable experts, (ii) education of the experts about the particular uncertainty characterization theory/approach selected for use, (iii) education of the experts about the analysis problem under consideration, (iv) development and implementation of mechanisms to assure that all available information is supplied to/shared by all experts, (v) development and implementation of procedures to assure that experts’ uncertainty characterizations are not unduly influenced by the opinions of other experts, (vi) development and implementation of appropriate information-harvesting (e.g. elicitation) procedures to obtain the experts characterizations of uncertainty expressed in the context of the particular uncertainty characterization theory/approach selected for use, and (vii) development and implementation of appropriate procedures for the aggregation of uncertainty results from multiple experts.

The complexity of designing and implementing an analysis that involves a separation of aleatory and epistemic uncertainty has been referred to before and is only touched on here as a reminder of its importance. In brief, this is a major conceptual and computational challenge. If this step is not carried out properly, the analysis could end up conceptually muddled, computationally impossible, and possibly both of the preceding.

Finally, there is the problem of communication of results. It does not matter how good the results of an analysis are if they are not understood or, even worse, if they are misunderstood and misused. This is a challenge in any analysis that involves a representation of the uncertainty in its results. Presentation of results is the easiest with interval analysis because of the simplicity of the concepts involved: analysis results are between a lower bound and an upper bound and that is that. After interval analysis, the presentation of results is probably the easiest with probability theory due to the familiarity with probability theory among the technically educated. However, even within this group, the understanding of probability and its use to represent epistemic uncertainty is not as great as one might expect, and misunderstandings and lack of understanding can be disconcertingly common. Communication of results obtained with evidence theory and possibility theory is likely to be more difficult and to require even greater educational efforts than is the case for probability theory. This difficulty results because few individuals (e.g. government project sponsors) are familiar with these approaches to the representation of uncertainty and thus will have no basis for the interpretation of the numbers that are being put before them. The preceding difficulties and challenges increase when an analysis that maintains a separation of aleatory and epistemic uncertainty is involved.

This presentation has focused exclusively on the representation and propagation of uncertainty. This topic is often referred to as uncertainty analysis. However, there is another important and closely related topic that deserves mention. This is sensitivity analysis, which refers to the determination of the importance of individual uncertain variables on the basis of how much they contribute to the uncertainty in analysis outcomes of interest. Sensitivity analysis has been extensively studied in the context of probabilistic representations of uncertainty [125–130], and a number of approaches to sensitivity analysis are available (e.g. differential analysis, response surface methodology, Monte Carlo analysis, Fourier amplitude sensitivity test, Sobol’ variance decomposition; see Ref. [68], Section 2, for a summary discussion of these techniques and references to additional sources of information). Many of the available techniques (e.g. examination of scatterplots, regression analysis, correlation analysis, rank transformations, searches for nonrandom patterns) involve an exploration
of a Monte Carlo mapping from model input to model results of the type used in the propagation of uncertainty in this presentation. How these results might be used for sensitivity analysis in the context of evidence theory, possibility theory and interval analysis certainly merits investigation. As another possibility, Soból’ variance decomposition [131–133] might provide a useful tool for sensitivity analysis in the context of evidence theory, with the incompletely specified probability spaces bounded by belief and plausibility giving rise to a range of variance decompositions for each uncertain variable.

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