# Bayesian forecasting and inference in latent structure for the Brazilian GDP and Industrial Production Index

GABRIEL HUERTA

*Department of Statistics, Northwestern University*

*2006 Sheridan Road. Evanston, Illinois 60208, U.S.A.*


and


HEDIBERT FREITAS LOPES

*DME, Federal University of Rio de Janeiro and*

*ISDS, Duke University*

*Caixa Postal 68530. Rio de Janeiro, RJ 21945-970, Brazil*

**Summary.** This paper considers the analysis of the Brazilian GDP and industrial production index using statistical tools recently developed for time series. The main purpose is the short-term forecast and structural decomposition of both series through an autoregressive model that allows, but not imposes, nonstationary behavior. A very strong point in this paper is that we incorporate all kinds of uncertainties by averaging forecasts across competing models, weighted by their posterior probability, in contrast with traditional analysis that puts probability one on a particular model.

*Some key words:* Autoregressive component models, Time series decomposition, Bayesian time series, Model uncertainty, Model averaging, Bayesian forecasting.

1

# 1   INTRODUCTION

Recently, many authors have shown interest in explaining and making short and long term predictions of the Brazilian GDP and industrial production index or some of its variants through univariate time series models, partially due to their importance in government policies and as the majors indicators of economical growth, and partially because such time series present nonstationary behavior, which makes the classical and Bayesian time series analysis considerably different. A few examples in the literature that present such issues are Lopes *et al.* (1998), Schmidt *et al.* (1998), Gamerman and Moreira (1998) and Cribari-Neto (1993). The first three papers apply in different ways dynamic linear models with structural components, such as local linear trends, seasonality and cycles, to describe the time behavior of the monthly observed industrial production index. Even though fairly general, their models do not account for uncertainty due to the specific choice of the trend/seasonal/cycle terms. In a rather different way Cribari-Neto (1993) analyses annually measured Brazilian GDP and shows that the cycle in that time series is relatively small when compared to its stochastic trend. As in the other referred papers, this one does not incorporate model uncertain when selecting the ARMA model for the first difference of the series. In contrast to what we present here, all of these papers report their results by selecting a particular model through some optimal criteria. In other words, the methods do not recognize the uncertainty involved in choosing a specific model, omission that can lead (and usually does) to extremely optimistic variances in forecasting.

In general, enormous amount of work has been done to develop useful models, but little has been devoted to incorporate model uncertainty as another issue in statistical modeling. In the context of time series, a classical work in this area is due to (Harrison and Stevens, 1976) that developed multi-process dynamic linear models in a attempt to broaden the class of models under consideration. More recently and not only related to time series, (Draper, 1995) suggests that model uncertainty should be taken very seriously both at forecasting levels and parameter estimation.

In this paper, we will use an approach that incorporates all uncertainties involved in time series modeling simultaneously. The model we use is Bayesian and allows for autoregressive unit roots so naturally avoids any of the corrections in significance tests proposed for the unitary root problem. For instance, an example of such corrections appears in Cribari-Neto (1993) that shows the complications that classical procedures induce to test if a root is unitary or not. Detailed discussion about unit root tests and their pitfalls is presented by (Campbell and Perron, 1991) and from a Bayesian viewpoint by (Uhlig, 1994). We have no purpose of suggesting that such works have no value but rather emphasize the unified approach offered from a Bayesian point of view to deal with model uncertainty and inference on autoregressive roots.

Specifically, for the analysis of the Brazilian GDP and industrial production index, we are using an autoregressive model with prior modeling on latent components and characteristic roots of the process as in (Huerta and West, 1998). The proposed model leads to a new class of prior distribution in autoregressive models which have the following properties:

- They permit arbitrary collections of real and complex conjugate pairs of characteristic roots.

- They allow for zero values among the characteristic roots, so taking care for prior uncertainty about model order.

- They allow unit roots, and so cater for persistent low frequency trends and sustained quasi-periodic components.

- They incorporate unobserved initial values of the data process as uncertain latent variables, so that all resulting inferences are formally based on incorporating full uncertainties about initial values.

It will be noticed on the following section, that a specific prior in the class is identified by a small number of hyper-parameters, which may be chosen based on specific forms of quantitative prior information. Alternatively (and usually), these hyper-parameters can assigned essentially uniform or "reference" prior distributions themselves, so inducing what may be viewed as a non-informative analysis. More on the advantages and implications of these prior specifications can be found in the seminal paper by (Huerta and West, 1998). We believe that an autoregressive model that recognizes full uncertainty on the order, model parameters and number of unitary roots can be very helpful to describe economics as that encompassed by the GDP or the industrial production index.

After a brief review of the time series models and methods in Section 2, we analyses the Brazilian industrial production index in section 3. Section 4 is reserved to the analysis of the Brazilian gross domestic product, while section 5 summarizes our findings and set our conclusions.

## 2  TIME SERIES MODEL AND METHODS

### 2.1  The Model and a Decomposition Result

Define $\{x_t\}$ as the realization of an autoregressive process of order $p$, $x_t = \phi(B)\epsilon_t$ where $Bx_t = x_{t-1}$ and $t \in \{0, 1, \ldots n\}$. $\phi(u) = 1 - \phi_1 u - \ldots - \phi_p u^p$ is the characteristic polynomial, $\phi = (\phi_1, \ldots, \phi_p)'$ is the vector of standard coefficients, and $\{\epsilon_t\}$ are zero-mean uncorrelated errors with $\epsilon_t \sim N(\epsilon_t|0, \sigma^2)$. Denote by $\{\alpha_1, \ldots, \alpha_p\}$ the reciprocals of the characteristic roots or solutions of the equation $\phi(u) = 0$. If $|\alpha_j| \leq 1$ for all $j$, the process is stationary

3

with unitary roots if any of these moduli equal one. Assume there are $C$ pairs of complex conjugate roots and $R = p - 2C$ real roots. Denote the complex pairs by $r_j \exp(\pm i\omega_j)$ for $j = 1, \ldots, C$, and the real roots by $r_j$ for $j = 2C + 1, \ldots, p,$.

As presented in West (1997), it can be shown that

$$x_t = \sum_{j=1}^{C} z_{tj} + \sum_{j=2C+1}^{p} a_{tj}$$

where the $z_{tj}$ and $a_{tj}$ are latent processes related to the complex and real roots respectively. Corresponding to the real roots $j = 2C + 1, \ldots, p$, the $a_{tj}$ are autoregressive processes of order one and corresponding to the complex conjugate pairs the $z_{tj}$ are autoregressive, moving average processes of order $(2, 1)$. That is, an autoregressive process can be expressed as the sum of simpler process some of periodic behavior and some with low frequency variation. In fact, the result implies that $z_{tj}$ has a quasi-periodic behavior with frequency $\omega_j$, or periodicity $\lambda_j = 2\pi/\omega_j$. The damping determined by the modulus of the defining complex root. Computation of the components can be handled through the state-space representation of the autoregressive model and has been exemplified in the context of an oxygen-isotope series in both West (1997) and West and Harrison (1997). In econometric applications, like this one we are proposing here, the latent process decomposition will shed some light on very important econometric issues, such as the probability that a particular root is unitary, or for how long a shock in $\epsilon_t$ will keep affecting $y_{t+h}$, for $h > 0$.

## 2.2 Prior Specifications

Huerta and West (1998) introduced a class of hierarchical priors on the component structure of autoregressive time series as just presented. We briefly review these specifications.

The prior assumes fixed but arbitrary upper bounds $C_+$ and $R_+$, on the number of complex pairs and real roots, hence an upper bound $p_+ = 2C_+ + R_+$ on model order. Independent priors are specified on the real roots, the complex roots and the innovations variance. Each real root $r_j$ has a prior that

- gives probability $\pi_{r,0}$ to $r_j = 0$,

- gives probability $\pi_{r,-1}$ to $r_j = -1$,

- gives probability $\pi_{r,1}$ to $r_j = 1$, and

- otherwise has a continuous density $g_r(r_j)$ from $-1$ to $1$.

Each complex conjugate pairs of roots $r_j \exp(\pm i\omega_j)$ has a prior that

- gives probability $\pi_{c,0}$ to $r_j = 0$,

4

- gives probability $\pi_{c,1}$ to $r_j = 1$, and

- otherwise has $r_j$ independent of $\omega_j$ with a continuous density $g_c(r_j)$ with support in $(0,1)$ for the modulus and $h(\lambda_j)$ a continuous density on the periods $\lambda_j = 2\pi/\omega_j$ with support on $(2, \lambda_u)$ and $\lambda_u$ is an upper bound. By default, $\lambda_u$ can be fixed to $n/2$ the maximum period observable in a time series of length $n$.

Notice that the prior is defined in the parameters that determine the time series decomposition of 2.1 so implicitly quantifies prior knowledge on latent structure. In previous applications of these priors particular forms for $g_r(\cdot)$, $g_c(\cdot)$ and $h(\cdot)$ have involved truncated Normals, Uniform densities or more general Beta distributions. A detailed exploration of how particular forms of these functions determine priors in other quantities of interest like the standard autoregressive coefficients, has been fully addressed in Huerta and West (1998). In this work and in a non-informative sense, we adopt the benchmark prior known as the *component reference prior* which implies that $g_r(\cdot)$ is a Uniform on $(-1,1)$, $g_c(\cdot)$ is a Beta(3,1) and $h(\lambda_j) \propto \sin(2\pi/\lambda_j)/\lambda_j^2$ with $\lambda_j$ ranging form 2 to $\lambda_u$. This is the standard reference prior obtained by treating components $z_{tj}$ and $a_{tj}$ individually.

Furthermore, the error variance is assumed independent of the roots and has a specified marginal prior, usually a conditionally conjugate inverse gamma prior. Priors for the point-masses may be specified as context dependent, but for simplification we use independent uniform Dirichlet distributions, namely $Dir(\pi_{r,0}, \pi_{r,1}, \pi_{r,-1}|1,1,1)$ and $Dir(\pi_{c,0}, \pi_{c,1}|1,1)$. Note that the prior point masses at zero for the numbers of roots, both complex and real, may fall below the fixed upper bounds. This implies that the model order can be anywhere from 0 to $p_+$. Also, the point masses at one allow for direct inferences on the number of unitary roots distinguishing between real and complex cases.

Additionally, we must note that the roots are not identified. The model is unchanged with arbitrary permutations of the roots. Identification of real roots can be imposed simply by relabeling them in order of increasing value. Identification can be similarly achieved for the complex roots by relabeling in order of increasing moduli or of increasing period or frequency.

## 2.3   Posterior and Predictive Analysis

Posterior and predictive calculations are developed using Markov chain Monte Carlo method based on the Gibbs sampling. With the *component reference prior* the conditional posterior distributions are easy to sample except for the complex pairs of roots. This updating of the complex roots requires a more complicated step based on the Metropolis-Hastings algorithm. For the interested reader, full details can be found in Huerta and West (1998).

# 3 ANALYZING THE BRAZILIAN INDUSTRIAL PRODUCTION INDEX

The main purpose of this paper is the analysis of two relevant macroeconomic time series from the Brazilian economy using AR models with priors on structure components and comparing to alternative models. First, we present the analysis of the *industrial production index* which consists of 215 observations measured on a monthly basis from February of 1980 to December of 1997. The data appears displayed on a time plot in Figure 1 from which two features must be noted: the strong cyclical-monthly pattern and the "ups" and "downs" driven by a trend. The series seems to present a non-stationary behavior.

We fitted an AR model, as described in Section 2, that allows for a maximal number of complex pairs of roots equal to 20 ($C_+ = 20$) and a maximal number of real roots also equal to 20 ($R_+ = 20$), allowing for a model order up to 60. The iterative algorithm to produce posterior samples was run for a burn-in of 5000 iterations, collecting the following 5000 samples but skipping every 50 iterations to break for potential autocorrelations induced in the MCMC. Based on this posterior samples, the Rao-Blackwell estimator of model order was obtained and appears in Figure 2; the posterior for $p$ is mostly concentrated within orders from 20 to 30 and a mode at around 24. Notice that the distribution is not favoring a few values for model order but instead reflects large uncertainty upon the lag of the autoregressive model. This pattern is not unique for the particular application and it has been noted in others, like those presented in the referred papers and enters in conflict with model selection via AIC or BIC.

Following with the analysis, Figure 3 shows the Rao-Blackwell estimator, based on the 5000 collected posterior samples, for the number of complex pairs of roots and number of real roots. Clearly the model likes 7 or 8 complex pairs with reasonably large probability values for 6 and 9 pairs. The posterior distribution for real roots is very disperse and centered around 10. This type of distributions have been obtained in other applications as well and usually reflects the existence of several components of very low frequency variation in the data. Figure 4 exhibits histograms of samples for the 2 smallest and two largest real roots when these are ordered in increasing value. Only those probabilities of point masses that are positive are reported in the picture. Two interesting issues here are the probability of .423 that the largest root (labeled $r(20)$) has of being unitary and the probability of 0.296 that the smallest root (labeled $r(19)$) has of being equal to $-1$. This confirms that data is indeed non-stationary with a possible random walk driving the trend of the series. We will show below that the component associated to the root that could be equal to 1, can be interpreted as this trend.

Posterior summaries for some of the complex roots appear in both Figures 5 and 6. They show boxplots of samples corresponding to the modulus and wavelength of 5 complex roots when, for identification, these are ordered by wavelength. The index "1" corresponds to the

root with larger period and so forth. The boxplots for moduli do not consider samples where the modulus equals to one and instead this posterior probability is reported in the lower right side of Figure 5. Observe that the root that has the larger period or wavelength has a posterior probability of being unitary equal to 0.77 and a period at around 12 time units. This complex root defines a quasi-cyclical non-stationary component that correspond to the seasonality in the data. The other roots also have a positive probability of being unitary with periods that are at about 6, 4, 3, and 2.4 units of time, respectively. Then, these roots have periodicities that can be interpreted as harmonics of the fundamental period of 12. Note that the fifth harmonic has a very high probability of its defining root being unitary if compared with the probabilities of the same event for the other roots. A very important feature of analyzing time series with AR models that allow for unitary roots and inference in quasi-cyclical patterns, is that the model is not imposing a particular type of seasonality or trend behavior but rather discovers the underlying structure in the data through a full Bayesian analysis with a hierarchical prior.

Posterior samples of the roots directly lead to samples for the components associated to the complex and real roots simply because these components are functions of the parameters in the AR model. In consequence, posterior summaries of the decomposition can be displayed as with other quantities of interest. In fact, Figure 7 presents the data with posterior means for two components corresponding to the complex roots and two components corresponding to the real roots. The quasi-cyclical component labeled by (C1) corresponds to that complex pair that has a periodicity of about 12 months and so can be interpreted as the underlying seasonality in the data. This component has a time-varying amplitude reflecting its non-stationary nature and this amplitude is comparable to the one presented by the series. Notice two high peaks between 1990-1992 that could reflect imposed economical policies by the country previous or in that time interval. Actually, during the years between 1986 and 1994 Brazil's economy has received numerous monetary shocks, such as the famous *Summer Plan* in 1986 or even the most recent *Real Plan*. The component labeled by (C2) corresponds to the root that has a harmonic periodicity of 6 months. It shows a very low amplitude in relation to the data and all other complex components have a similar pattern in terms of this characteristic. Furthermore, the component labeled by (R3) corresponds to the maximal real root, that as shown in Figure 4 has about 2/5 probability of being unitary. This component has a similar amplitude as the data and we think that drives its underlying trend. Once again, the changes in the Brazilian economy mentioned previously can be seen in this components as well. During the period 1988-1992, this components looks like a legitimate random walk with larger ups and downs. The last posterior mean displayed (R4) corresponds to the smallest real root; its amplitude is very low in regards to the data and presents this switches in time particular of AR(1) processes that have a root

7

equal or close to $-1$.

We must establish at this point that the model was fitted with all the observations previous to and including January 1997. The remaining 11 observations were left out for model validation and forecasting. As with exploration for the posterior distribution of model parameters, inference on future observations can be obtain via simulation and as an additional step in the Gibbs sampler. Samples of forecasts can be generated conditional on all other parameters using the autoregressive equation that defines the model in a recursive manner. Based on 5000 of these posterior samples, Figure 8 presents the 95% predictive probability intervals and posterior means for forecasts corresponding to February 1997 to December 1997 compared against the actual observed values. Posterior means are lower but close to the observed values.

In terms of comparison, we computed similar forecasts with AR models, model order selected using the AIC and for two different situations. One in which we produce paths of forecasts for the period of interest treating the maximum likelihood estimator of the model parameters as the "true" parameters. In the second framework, we assume that the standard model coefficients $\phi$ and error variance $\sigma^2$ has the reference prior $p(\phi, \sigma^2) \propto 1/\sigma^2$. The reference posterior is a Normal-Gamma with posterior mean equal to the maximum likelihood estimator. This distribution can be easily sampled and so the corresponding forecasts under this Normal-conjugate setting. For the Brazilian IPI, AIC leads to a model order of 13, in the lower tail of the posterior distribution of Figure 2. In fact, Figure 9 compares the 95% predictive forecasts from February 1997 to December of the same year with the 3 models discussed so far: the AR with priors on structure components, an AR model treating point estimators as true parameters and an AR model with a reference prior on $\phi$ and $\sigma^2$. Intervals for the AR model with priors on the roots have more or less the same width as those corresponding to the other models. Also, we note that the intervals are shifted down with respect to those obtained with structured priors on roots and will tend to underestimate more the observed values. Actually, using the posterior means of forecasts as point estimators of future values, it results that the mean square error (MSE) with our model is 32.6781 and for the models that use AIC, the MSE is 45.62093 with the standard reference prior and 61.1393 with the maximum likelihood estimator. This application confirms that when AR models are used as empirical devices for data, it is worth the effort of incorporating model uncertainty, unitary roots and treatment of initial values in a scenario that requires forecasting.

Other models have been used to forecast some of the months of the industrial production index for 1997. Once again, we refer here to the analysis of this series presented in Schmidt *et al.* (1998) and Gamerman and Moreira (1998) through Dynamic Linear Models. A comparison of point estimators of forecasts for March 1997 to August 1997 obtained

with this dynamic models and AR models as discussed in this paper, appears in Figure 10. Evidently, all the five models considered underestimate the observed values with dynamic models having a better performance than autoregressive processes. This is not a surprising, since Dynamic Linear Models are usually more adequate for short-term forecasting of nonstationary series than AR or more general ARMA models. On the other hand, the AR model with priors on roots do not required the specification of particular trend, seasonal components or differentiation of the series. In terms of predictive intervals, Figure 11 compares the 95% probability intervals for the AR models with priors on component structure and the dynamic models of Schmidt *et al.* (1998) and Gamerman and Moreira (1998), again for the period that covers March 1997 to August 1997. We find interesting that the AR model contains the predictive intervals for the Dynamic Models except for June 1997. This leads to the conclusion that incorporating model uncertainty produces more conservative predictive intervals, which will eventually be used by policy makers as a measure of risk for their alternative scenarios.

## 4 ANALYZING THE BRAZILIAN GDP

In this section, we emphasize on the analysis of the Brazilian GDP series. The data, displayed in 12 in logarithmic scale, consists of 91 annual observations of the GDP since the beginning of the century. Since the seasonality has been removed, the data has no obvious cyclical patterns and only seems driven by a trend. This data has been previously analyzed in Cribari-Neto (1993) where the standard and augmented Dickey-Fuller tests and the Phillips-Perron test for unitary roots were applied to the series. The author concluded that even at the 10% level it is not possible to reject the null of a unit root and hence the series has a stochastic trend. After reaching this conclusion, the series was differenced once and an ARMA model fitted via maximum likelihood with orders on the AR and MA parts chosen with the bias-corrected version of AIC. Cribari-Neto finds that an AR(1) with coefficient 0.0653 is adequate to the differenced series (corrected by the mean) and this implies an AR(2) model for the logarithm of the GDP with a couple of real roots, one of them unitary. Now, we discuss the results of an analysis of the series using AR models with prior on structure components.

Since cycles are not expected in the data, we fitted an AR that has no complex pairs of roots ($C_+ = 0$) and allows for a maximal number of real roots $R_+$, equal to 15. Other analysis with $C_+ > 0$ showed that this is a sensible specification since most of the mass for the posterior distribution of number of complex pairs is concentrated at zero. We obtained 5000 posterior samples after an initial burn-in of 5000 iterations and skipping every 50 for recollection. Figure 13 presents the posterior distribution for model order based on these samples. Notice that there is probability one for the model order to be greater than 2, most

9

of the mass is equally concentrated at 3 and 4 with decaying probability up to an order of 10. Following with the analysis, in Figure 14 we present histograms of samples for the smallest and two largest real roots of the model. There is a very high probability (0.9425) that the largest root is unitary, consistent with Cribari-Neto's analysis, but also there is a slight chance (0.087) of the existence of a second unitary root. In regards to the point mass at zero for each root, only the smallest and the two largest have probability one of not being null.

Applying the decomposition result of Section 2, we computed samples of those components corresponding to the minimal and the two maximal roots, which are the only ones that have no probability of being zero. The samples were obtained conditional on only one unitary root. That is, we discarded those samples where we had two unit roots, since in this case the decomposition is not applicable. The posterior mean of the sum of the three components is presented in Figure 12 with the data. We believe that this mean can be interpreted as the underlying stochastic trend in the logarithm of the GDP.

## 5  CONCLUSIONS

This paper analyzes the Brazilian industrial production index and GDP using a Bayesian methodology based on a new class of prior distributions for AR models. The two applications discussed show how a unified approach is able to deal with model uncertainty, inference on latent structure, perhaps of quasi-cyclical nature, inference on unitary roots or stochastic trends and forecasting, all simultaneously. It avoids the imposition of trends and polynomial seasonal components to capture structure and multiple significant tests to show the presence of an underlying stochastic trend.

## ACKNOWLEDGEMENTS

## Bibliography

Campbell, J. Y. and Perron, P. (1991) Pitfalls and opportunities: What macroeconomists should know about unit roots. NBER-Working paper series 100.

Cribari-Neto, F. (1993) The cyclical component in brazilian gdp. *Revista de Econometria*, **13,** 1–22.

Draper, D. (1995) Assessment and propagation of model uncertainty (with discussion). *Journal of the Royal Statistical Society-Series* B, **57,** 45–97.

Gamerman, D. and Moreira, A. R. B. (1998) Bayesian analysis of econometric time series models using hybrid integration rules. *Journal of Econometrics (in review)*.

Harrison, P.J. and Stevens, C.F. (1976) Bayesian forecasting (with discussion). *Journal of the Royal Statistical Society-Series* B, **38,** 205–247.

Huerta, G. and West, M. (1998) Priors and component structures in autoregressive time series models. *Journal of the Royal Statistical Society-Series* B, **61,** 1–19.

Lopes, H. F., Moreira, A. R. B. and Schmidt, A. M. (1998) Hyperparameter estimation in forecast models. *Computation Statistics and Data Analysis (to appear)*.

Schmidt, A. M., Gamerman, D. and Moreira, A. R. B. (1998) An adaptive resampling scheme for cycle estimation. *Journal of Applied Statistics (in review)*.

Uhlig, H. (1994) What macroeconomists should know about unit roots: A Bayesian perspective. *Econometric Theory*, **10,** 645–671.

West, M. (1997) Time series decomposition. *Biometrika*, **84,** 489–494.

West, M. and Harrison, J. (1997) *Bayesian Forecasting and Dynamic Linear Models (2nd Edn.)*. New York: Springer-Verlag.
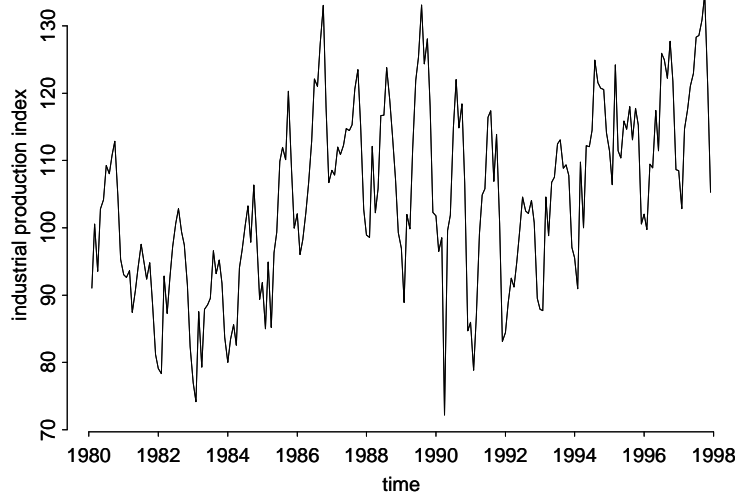
Figure 1: *Brazilian Industrial Production Index.* 215 Monthly observations taken since February 1980.
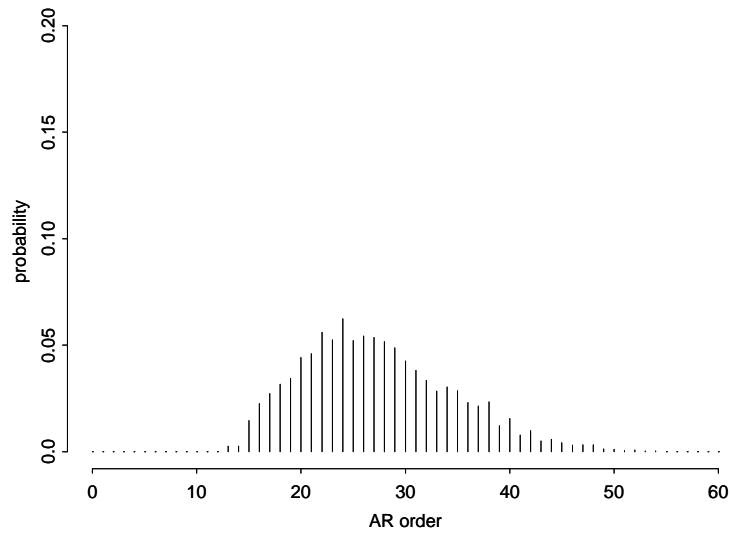


Figure 2: *IPI analysis.* Posterior distribution for model order $p$ based on 5000 posterior samples; $C_+ = 20$ and $R_+ = 20$
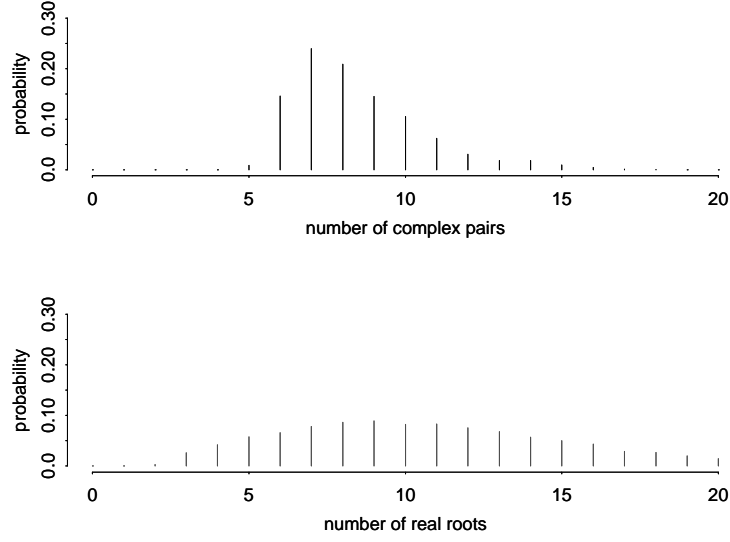
12

Figure 3: *IPI analysis.* Posterior distributions for number of complex pairs and number of real roots based on 5000 posterior samples; $C_+ = 20$ and $R_+ = 20$
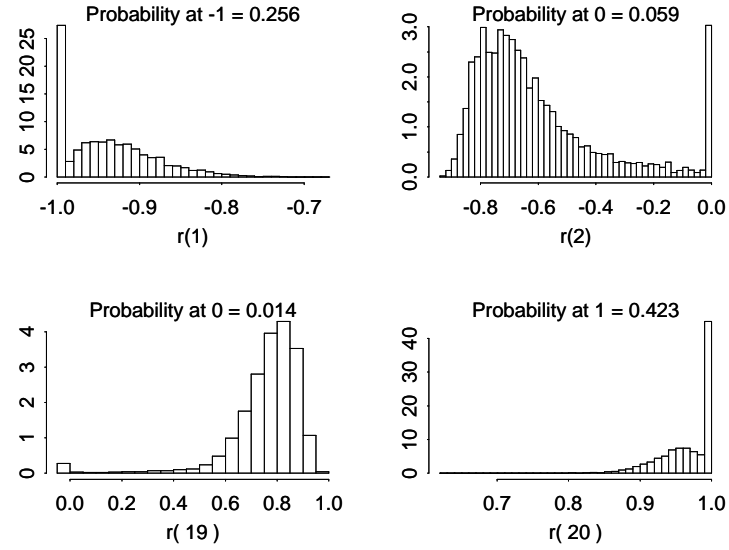


Figure 4: *IPI analysis.* Histograms of samples for the two largest and the two smallest real roots with reported posterior probabilities of point masses if they are positive.
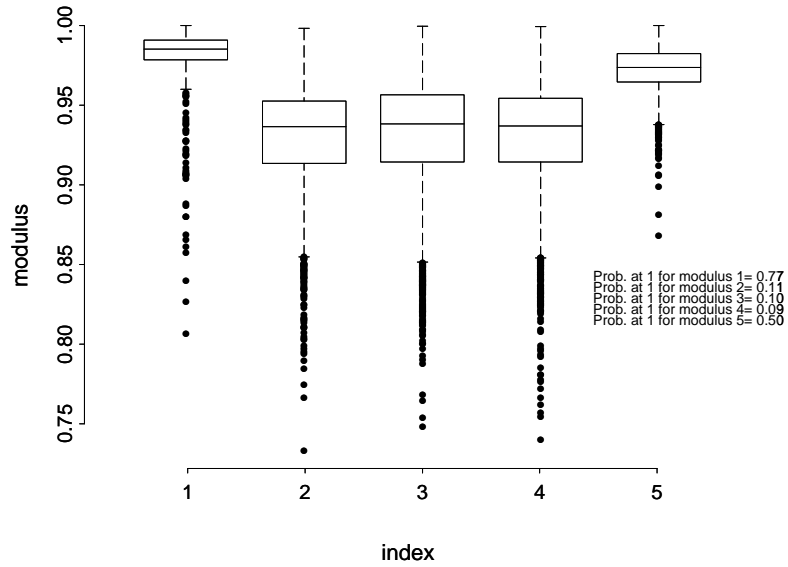
Figure 5: *IPI analysis.* Boxplots of samples for moduli corresponding to the 5 largest roots ordered by wavelength with reported posterior probability of a point mass at one.
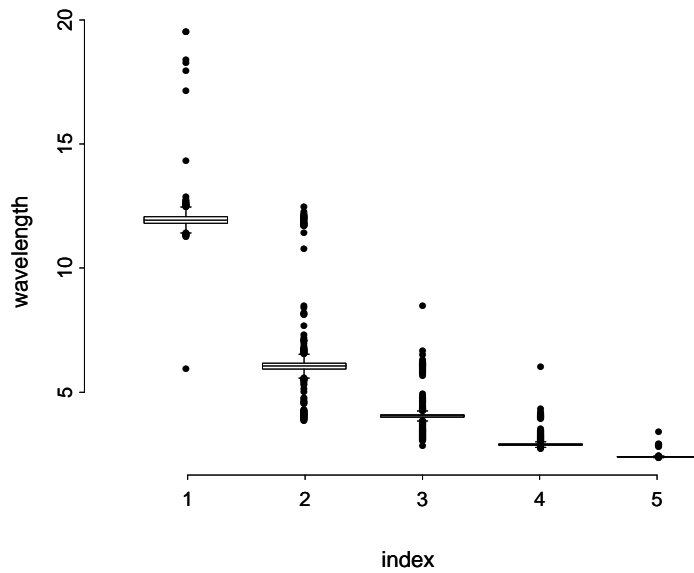


Figure 6: *IPI analysis.* Boxplots of samples for wavelengths corresponding to the 5 largest roots ordered by wavelength.
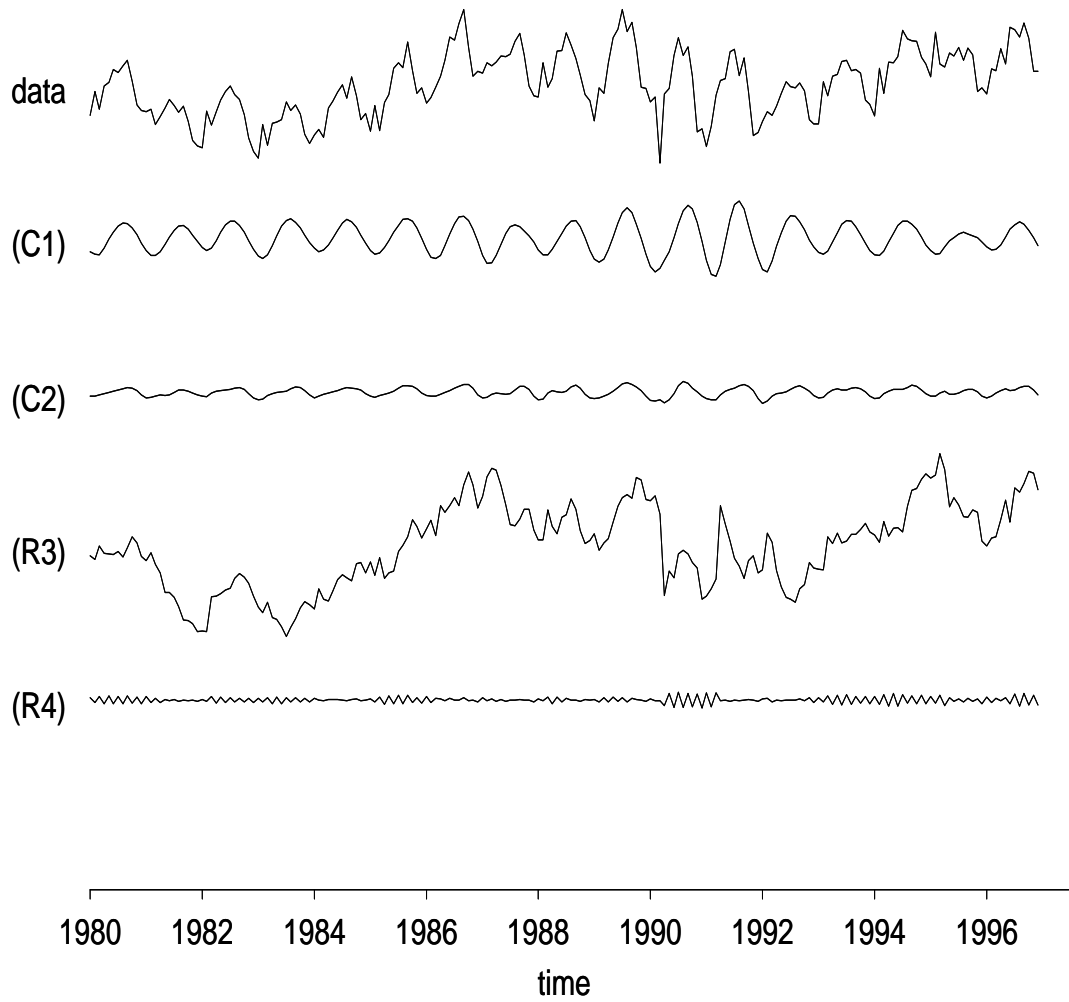
Figure 7: *IPI analysis* Data and posterior means for two latent components corresponding to complex roots and two latent components corresponding to real roots. Complex components (labeled C1 and C2) are the two maximal when ordered by wavelength and real components (labeled R3 and R4) are the corresponding to the maximal and minimal roots.
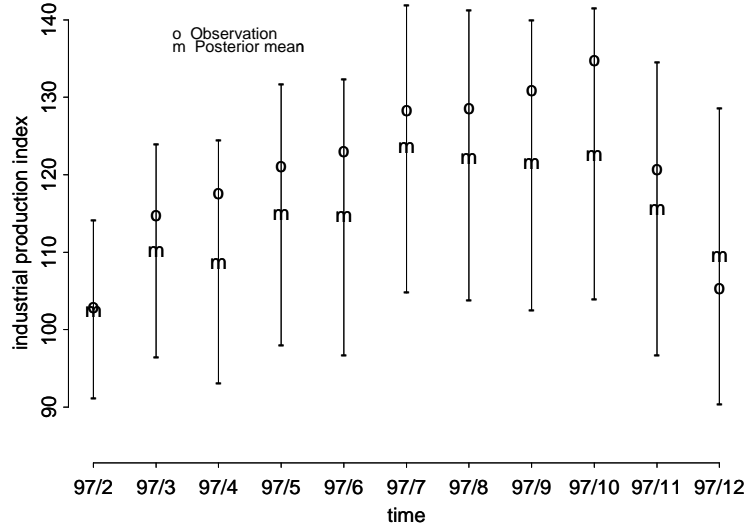
Figure 8: *IPI analysis.* 95 % predictive intervals based on AR model with priors on structure components including posterior means and actual observations for 1997.
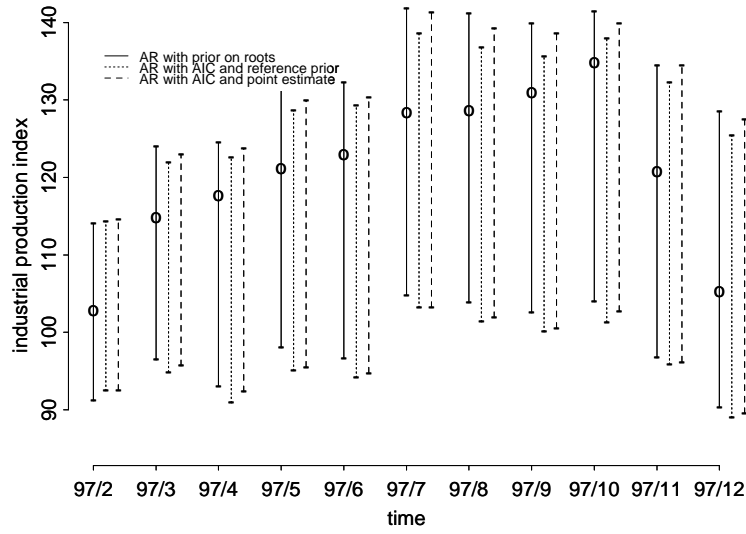


Figure 9: *IPI analysis.* 95 % predictive intervals based on AR models with priors on structure components, order selected via AIC and a reference prior on AR coefficients and order selected via AIC and MLE. "o" marks an actual observation.
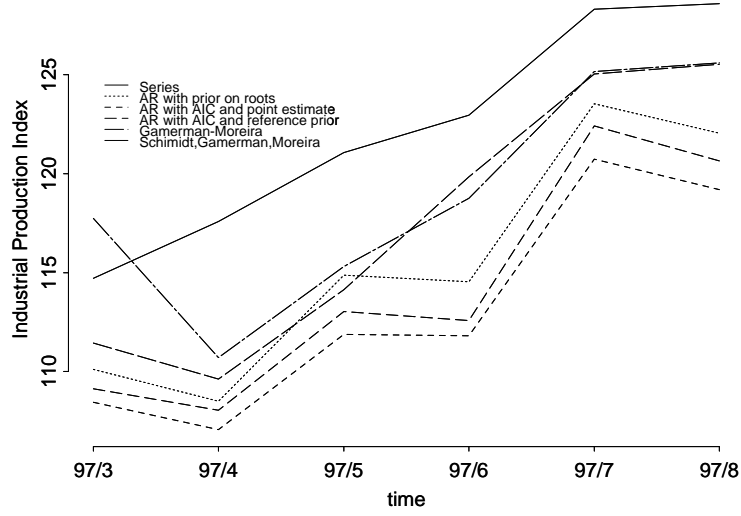
Figure 10: *IPI analysis.* Observations and forecasts for 6 months, 97/3 to 97/8, corresponding to five time series models.
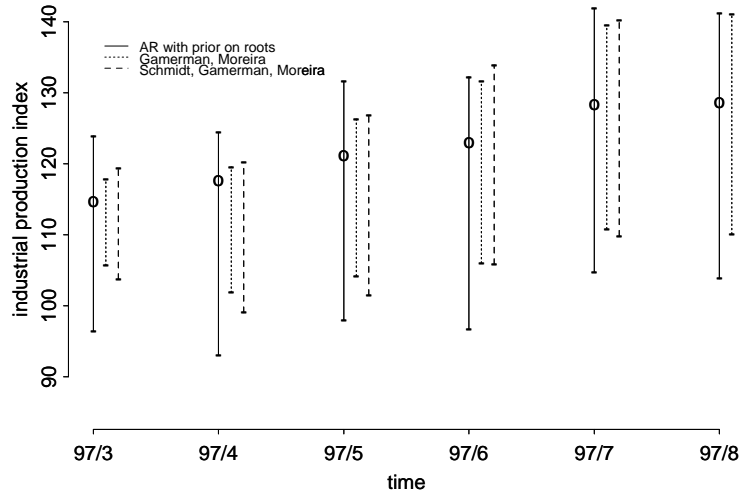


Figure 11: *IPI analysis.* 95 % predictive intervals for the period 97/3-97/8 based on AR models with priors on structure components, Dynamic Linear Models as in Gamerman-Moreira and Dynamic Linear Models as in Schmidt, Gamerman and Moreira
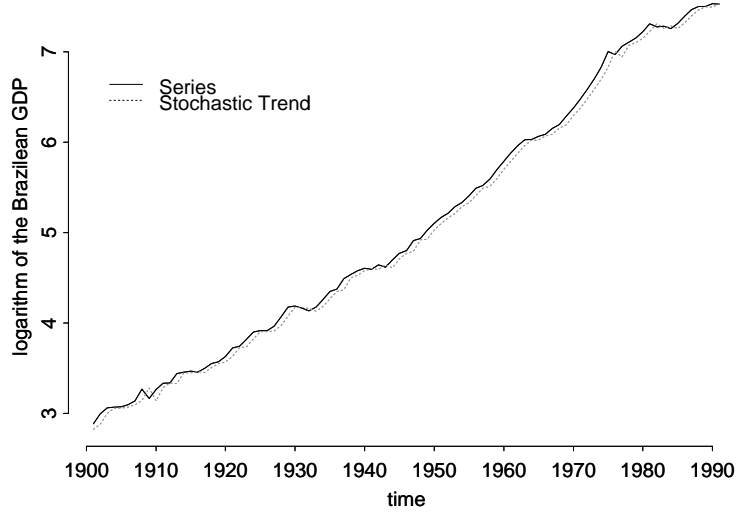
Figure 12: *GDP analysis* Logarithm of the series (solid line) and posterior mean for the stochastic trend (dotted line) based on an AR model with $C_+ = 0$ and $R_+ = 15$. Data was annually observed over a period of 91 years.
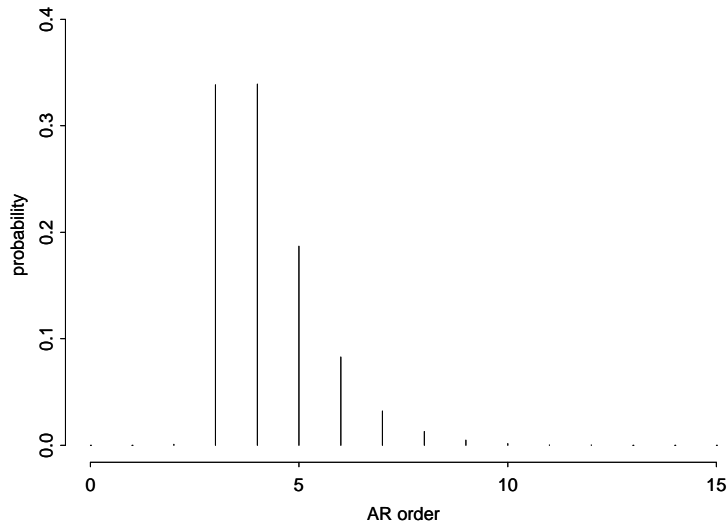


Figure 13: *GDP analysis* Posterior distribution for model order $p$ based on 5000 posterior samples; $C_+ = 0$ and $R_+ = 15$.
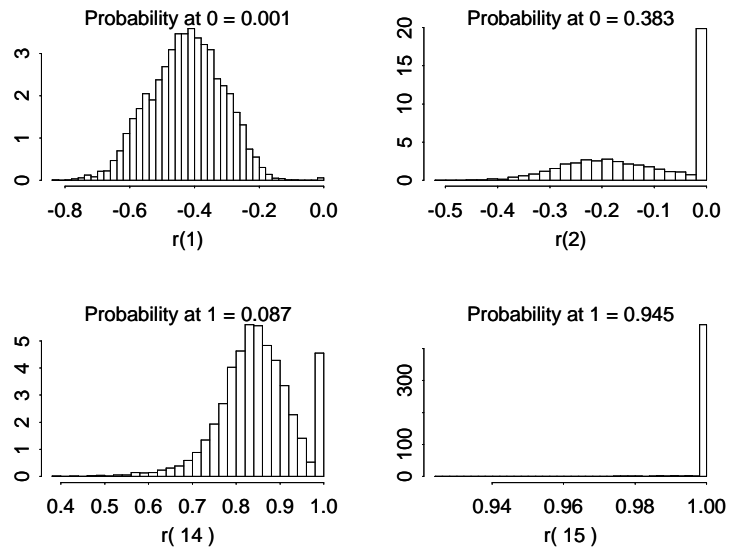
18

Figure 14: *GDP analysis.* Histograms of samples for the two largest and the two smallest roots with reported probabilities of point masses if positive.