Statistical Inference 453/553. Spring 2003

Solutions HW2. Selected Problems

February 26, 2002

1. **Exercise 6.6** If $X \sim Gamma(\alpha, \beta)$ then the joint pdf of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is:

$$f(\mathbf{x}|\alpha, \beta) = \frac{1}{\Gamma(\alpha)^n \beta^n} (\prod_{i=1}^{n} x_i)^{\alpha-1} exp(-\sum_{i=1}^{n} x_i/\beta)$$

By the factorization theorem, $T = (\prod_{i=1}^n X_i, \sum_{i=1}^n X_i)$ is a two-dimensional sufficient statistic for (α, β) (BTW, this is also a minimal sufficient statistic).

2. **Exercise 6.12** (a) This item was discussed in class and through an e-mail note. (b) $E(X/N) = E\left(E(X/N|N=n)\right) = (1/n)E(E(X|N=n)) = (1/n)E(n\theta) = \theta$;

$Var(X/N) = E(Var(X/N|N=n)) + Var(E(X/N|N=n)) = E(\theta(1-\theta)/n) + Var(n\theta/n) = \theta(1-\theta)E(1/N) + 0 = \theta(1-\theta)E(1/N)$.

3. **Exercise 6.15** (a) The pair $(\theta, a\theta^2)$ defines a parabola on R^2. Then, it does not contain a two-dimensional open set.

(b) Check your class notes.

4. **Exercise 6.18**. Since the $X's$ follow a Poisson(λ) distribution, $T = \sum_{i=1}^n X_i$ has a Poisson($n\lambda$) distribution. If $g(T)$ is any function of the statistic T then,

$$E(g(T)) = \sum_{t=0}^{\infty} g(t) \frac{(n\lambda)^t e^{-n\lambda}}{t!}$$

If $E(g(T)) = 0$ then $\sum_{t=0}^{\infty} g(t)\frac{(n\lambda)^t}{t!} = 0$ for all values of λ and n. Since n and λ are strictly greater than zero, the last expression implies that $g(t) = 0$ for $t = 0, 1, 2, \ldots$. Then, T is a complete statistic for the Poisson($n\lambda$) family of distributions.

5. **Exercise 6.21** (a) No. Simply consider $g(X) = X$, then $E(X) = -1(\theta/2) + 0(1-\theta) + 1(\theta/2) = 0$ but $T = X$ is not identically zero with probability one.

(b) Let $g(|X|)$ be any function of $|X|$, then

$$E(g|X|) = g(|-1|)(\theta/2) + g(|0|)(1-\theta) + g(|1|)(\theta/2) = g(|1|)(\theta) + g(|0|)(1-\theta)$$

From this expression, we have that $E(g(|X|)) = 0$ if and only if $g(|1|) = 0$ and $g(|0|) = 0$. Also, notice that $g(|1|) = g(|-1|)$. Then $g(|X|) = 0$ with probability one.

(c)We have that $f(x|\theta)$ can be expressed as:

$$f(x|\theta) = 2^{-|x|}(1-\theta)exp(|x|log(\theta/(1-\theta)))$$

which has the form of an exponential family model.

6. **Exercise 6.30** (a) The joint pdf of $X_1, \ldots X_n$ is

$$f(x_1, x_2, \ldots, x_n|\mu) = exp(-\sum_{i=1}^{n}(x_i - \mu))I_{(-\infty, X_{(1)})}(\mu) = exp(-\sum_{i=1}^{n} x_i)exp(n\mu)I_{(-\infty, X_{(1)})}(\mu)$$

Then, by the factorization theorem, $X_{(1)}$ is a sufficient statistic for μ.

If $Z = X_{(1)}$, the pdf of Z is $f(z|\mu) = n\ exp(-n(z-\mu))I_{(\mu,\infty)}(z)$. Let $g(Z)$ be any function of Z, if we make

$$E(g(Z)) = \int_{\mu}^{\infty} g(z)f(z|\mu)dz = 0$$

then,

$$\int_{\mu}^{\infty} g(z)exp(-nz)dz = 0$$

For this last expression, if we take the derivative with respect to μ at both sides, we get the equation:

$$-g(\mu)exp(-n\mu) = 0$$

Since this expression is valid for any quantity $\mu > 0$ and $exp(\cdot) > 0$ then $g(\mu) = 0$ for any value of μ. $g(Z) = 0$ with probability one.

(b) To use Basu's theorem, we need to show that S^2 is ancillary. Realize that for value of $i = 1, \ldots, n$, $Z_i = X_i - \mu$ has an $Exp(1)$ distribution. The other thing is to note that $(X_i - \bar{X}) = ((X_i - \mu) - (\bar{X} - \mu)) = (Z_i - \bar{Z})$. Then

$$S^2 = \sum_{\frac{1}{n-1}} \sum_{i=1}^{n}(Z_i - \bar{Z})^2$$

Since S^2 depends on the random variables $Z's$, its distribution is inherited by the joint pdf of $Z_1, Z_2, \ldots, Z_n$ which does not depend on μ. Then, S^2 is ancillary.

7. **Exercise 6.34**

The likelihood for a sample point (n, x) as in Exercise 6.12 is:

$$L(\theta|n, x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}p_n$$

The likelihood for a point x in the fixed-sample-size Binomial experiment is:

$$L(\theta|x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

Then the ratio,

$$\frac{L(\theta|n,x)}{L(\theta|x)} = p_n$$

so the likelihoods are proportional. By the likelihood principle, the conclusions about θ are the same with N fixed or random.

8. **Exercise 6.36** (a) By definition, $E(U_1|T_2) = E(E(U|T_1)|T_2)$. Since $E(X) = E(E(X|Y))$, then $E(E(U|T_1)|T_2) = E(U|T_1)$. By definition of minimal sufficient statistic, we know that T_2 is function of T_1, so taking conditional expectation given T_1 is the same as taking conditional expectation given T_2. Hence, $E(U|T_1) = E(U|T_2) = U_2$

   (b) In general, we know that $Var(X) = E(Var(X|Y)) + Var(E(X|Y))$. Then, $Var(U_1) = E(Var(U_1|T_2)) + Var(E(U_1|T_2))$. Using part(a), we have that

$$Var(U_1) = E(Var(U_1|T_2)) + Var(U_2) \geq Var(U_2)$$

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